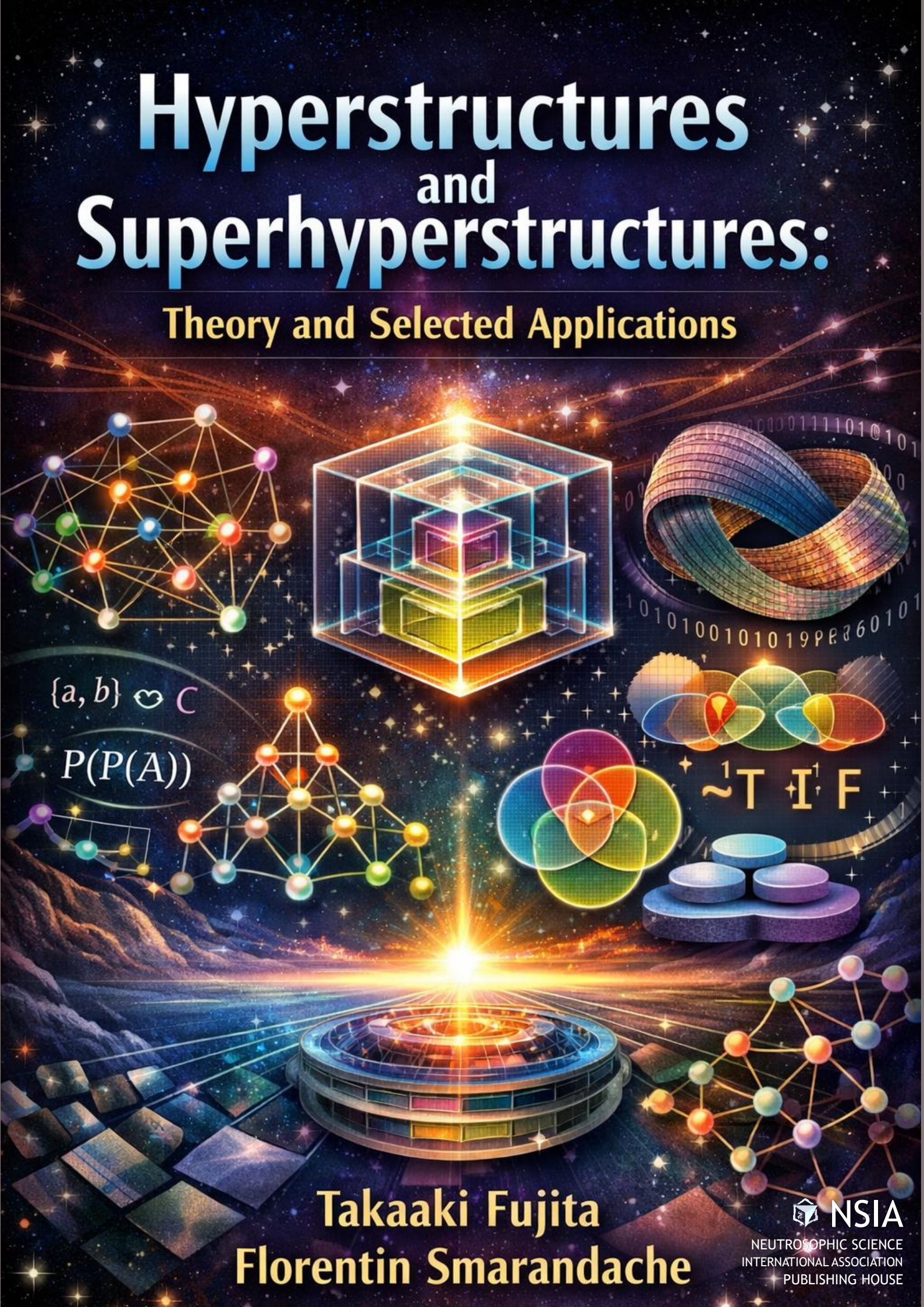
Hyperstructures and Superhyperstructures:

Theory and Selected Applications



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Abstract

In this book, the term *structure* is used in a broad sense: it covers both abstract mathematical objects (such as graphs, topological spaces, algebras, automata, or games) and idealized representations of real–world systems. A *hyperstructure* relaxes the usual notion of an algebraic structure by allowing an operation to return an entire *subset* of possible outcomes instead of a single element. By iterating the powerset construction n times one obtains the iterated powerset $\mathcal{P}_n(H)$, and operations acting on these objects lead to the notion of an *n-superhyperstructure*.

This book conducts a dynamic survey of the various notions defined within this general structural framework. We hope that the present perspective will help readers apply hyperstructural and superhyperstructural ideas to a wide range of complex hierarchical phenomena, and that it will provide fertile ground for discovering further new concepts of HyperStructure and SuperHyperStructure.

Keywords: HyperStructure, SuperHyperStructure, HyperAlgebra, HyperGraph

Chapter 1

Introduction

1.1 HyperStructure and SuperHyperStructure

In this book, the term *Structure* is used in a broad sense: it covers both abstract mathematical objects (such as graphs, topological spaces, algebras, automata, or games) and idealized representations of real–world systems. Many of these systems possess an intrinsically hierarchical organization, and for such situations the formalisms of *hyperstructures* and *superhyperstructures* provide a convenient and expressive framework.

A *hyperstructure* relaxes the usual notion of an algebraic structure by allowing a binary operation to return an entire *subset* of possible outcomes instead of a single element (see, e.g., [1–4]). This point of view has motivated many hyperstructure–based models, including HyperGraphs [5], HyperFuzzy Sets [6,7], HyperNeutrosophic Sets [8], HyperAlgebras [9,10], HyperGames [11,12], and HyperVectors [13], among others, all of which are closely connected to the hyperstructure paradigm.

By iterating the powerset construction n times one obtains the iterated powerset $\mathcal{P}_n(H)$, and operations acting on these objects lead to the notion of an *n-superhyperstructure* (see, for example, [1,14]). Such n-superhyperstructures are intended to model complex, multi–layered phenomena in a transparent way, by organizing interactions across several levels of aggregation. In analogy with hyperstructures, a number of superhyperstructure–based concepts have been proposed, including SuperHyperGraphs [15–17], SuperHyperFuzzy Sets [18,19], and SuperHyperAlgebras [20,21].

For later reference, Table 1.1 summarizes the basic variants that will appear in this work.

Remark 1.1.1 (Convention on the profile (m,n,k))**.** Throughout this book, we interpret a profile $\Theta = (m,n,k)$ as follows: m specifies the *input level* (the arguments live in $\mathcal{P}^m(H)$), n specifies the *output level* (values lie in $\mathcal{P}^n(H)$), and k is the *arity* (the number of inputs combined). Thus a Θ–superhyperoperation has the type

$$\star : \big(\mathcal{P}^m(H)\big)^k \longrightarrow \mathcal{P}^n(H),$$

where larger m and n correspond to deeper nested (multi-level) representations.

Table 1.1: Classical structures, hyperstructures, superhyperstructures, and (m,n)-superhyperstructures: underlying domains and typical operation signatures

Structure type	**Underlying domain**	**Typical operation signature**
Classical structure	H	$\#_0 : H^m \to H$
Hyperstructure	$\mathcal{P}(H)$	$\circ : H \times H \to \mathcal{P}(H)$
n-superhyperstructure	$\mathcal{P}_n(H)$	$\circ : \mathcal{P}_n(H) \times \mathcal{P}_n(H) \to \mathcal{P}_n(H)$
(m,n)-superhyperstructure	$\mathcal{P}^m(H)$	$\star : \left(\mathcal{P}^m(H)\right)^k \to \mathcal{P}^n(H)$

Notation. For a set H, its powerset is $\mathcal{P}(H)$. The iterated powersets are given by $\mathcal{P}_0(H) = H$ and $\mathcal{P}_{n+1}(H) = \mathcal{P}(\mathcal{P}_n(H))$ for $n \geq 0$, and we write $\mathcal{P}^m(H) = \mathcal{P}_m(H)$ when no confusion arises. If the empty set is to be excluded, we use $\mathcal{P}_*^n(H) = \mathcal{P}^n(H) \setminus \{\emptyset\}$. As usual, H^m denotes the m-fold Cartesian product, and k is the arity of the operation.

HyperStructure and SuperHyperStructure are worth studying for the following reasons. Research on HyperStructure and SuperHyperStructure is highly important.

- **Higher–order interactions:** model genuinely multiway relations.
- **Non-determinism:** encode one-to-many outcomes via set-valued operations.
- **Uncertainty:** unify fuzzy/neutrosophic/rough-style semantics in one language.
- **Hierarchy (superhyper):** capture nested multi-level structure by $\mathcal{P}^n(\cdot)$.
- **Backward compatibility:** recover classical structures via $S \hookrightarrow \mathcal{P}^n(S)$.
- **Systematic theory:** develop morphisms, dynamics, and categorical constructions.
- **Applications:** support set-based algorithms in AI, optimization, and decision-making.

1.2 Our Contributions

For these reasons, research on HyperStructures and SuperHyperStructures is important. However, survey-type references on these topics are still relatively scarce. To help bridge this gap, this book provides a broad survey of the notions and modeling principles that arise in the hyperstructural and superhyperstructural framework. In particular, we systematize (i) hyperstructures based on set-valued operations, (ii) superhyperstructural extensions built on iterated powersets and hierarchical incidence, and (iii) representative variants and extensions that appear across different mathematical and applied contexts.

Along the way, we assemble core definitions, illustrative examples, and recurring construction patterns, and we organize a wide range of themes—including uncertainty-aware generalizations (e.g., fuzzy- and set-valued semantics), multi-level and higher-order relational modeling, and locality/gluing viewpoints (e.g., sheaf-like constructions)—within a unified set-theoretic notation. We hope that this perspective will help readers translate complex hierarchical phenomena into hyperstructural or superhyperstructural models, compare alternative formalisms on a common footing, and identify natural directions for further extensions and new concepts.

Chapter 2

Basic Concepts

In this chapter we fix notation and recall the basic notions that will be used throughout the book.

2.1 Hyperstructures and Superhyperstructures

Many mathematical models and real–world systems naturally exhibit several hierarchical layers (for instance, elements, groups of elements, and groups of groups). To describe such multi–level interactions in a uniform way, one often works with *hyperstructures* and their iterated variants, called *superhyperstructures*. Informally, hyperstructures allow operations to return *subsets* rather than single elements, and superhyperstructures iterate this idea to operate on *nested* collections (sets of sets, sets of sets of sets, etc.) [3, 22]. As related concepts to HyperStructures, weak hyperstructures are also known [23–26].

Definition 2.1.1 (Base set)**.** [27] A *base set* (or *ground set*) is a fixed nonempty set S. All constructions below (powersets, iterated powersets, and operations acting on them) are formed from S.

Definition 2.1.2 (Powerset)**.** [28] The *powerset* of a set S, denoted by $\mathcal{P}(S)$, is the set of all subsets of S:

$$\mathcal{P}(S) = \{\, A \mid A \subseteq S \,\}.$$

In particular, $\emptyset \in \mathcal{P}(S)$ and $S \in \mathcal{P}(S)$. For convenience, we also write

$$\mathcal{P}_*(S) := \mathcal{P}(S) \setminus \{\emptyset\}$$

for the family of all *nonempty* subsets of S.

Definition 2.1.3 (Hyperoperation)**.** (cf. [29, 30]) Let S be a nonempty set. A *hyperoperation* on S is a map

$$\circ : S \times S \longrightarrow \mathcal{P}(S).$$

If, moreover, $\circ(x, y) \neq \emptyset$ for all $x, y \in S$, then $\circ$ is called a *classical-type* hyperoperation and we may view it as $\circ : S \times S \to \mathcal{P}_*(S)$. Allowing $\circ(x, y) = \emptyset$ (for some inputs) is sometimes referred to as a *neutrosophic-type* convention in applications.

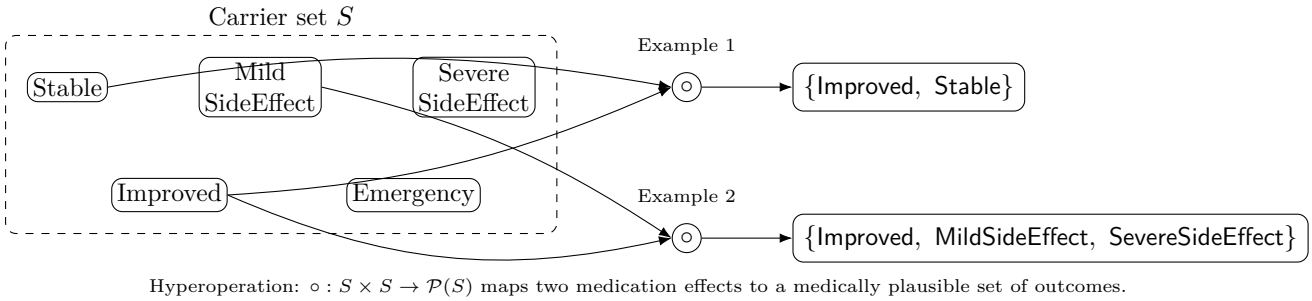


Figure 2.1: Schematic overview of the drug–drug interaction hyperstructure $(S, \circ)$ with two illustrative outcomes.

Definition 2.1.4 (Induced operation on subsets)**.** Let $\circ : S \times S \to \mathcal{P}(S)$ be a hyperoperation. Its canonical extension to subsets is the map

$$\odot : \mathcal{P}(S) \times \mathcal{P}(S) \longrightarrow \mathcal{P}(S), \qquad A \odot B := \bigcup_{a \in A} \bigcup_{b \in B} (a \circ b).$$

(We use the convention that a union over an empty index set is $\emptyset$.) If $\circ$ is classical-type and $A, B \in \mathcal{P}_*(S)$, then $A \odot B \in \mathcal{P}_*(S)$.

Definition 2.1.5 (Hyperstructure)**.** (cf. [1, 23, 31, 32]) Let S be a nonempty set and let $\circ : S \times S \to \mathcal{P}(S)$ be a hyperoperation. The pair $(S, \circ)$ is called a *hyperstructure* (more precisely, a *hypergroupoid*). Via Definition 2.1.4, it induces the *powerset-level* structure $(\mathcal{P}(S), \odot)$, where $\odot$ combines collections of elements into new collections.

Example 2.1.6 (Drug–drug interaction outcomes as a hyperstructure)**.** *Interpretation.* Let S be a set of clinically relevant patient states after medication, e.g.,

$$S = \{\mathsf{Stable},\ \mathsf{MildSideEffect},\ \mathsf{SevereSideEffect},\ \mathsf{Improved},\ \mathsf{Emergency}\}.$$

Different patients (genetics, comorbidities, dosage timing) may exhibit different outcomes when two drugs are co-administered.

Hyperoperation. For $x, y \in S$, define $x \circ y \subseteq S$ to be the set of all medically plausible resulting states when the two medication effects corresponding to x and y occur together. Then

$$\circ : S \times S \longrightarrow \mathcal{P}(S), \qquad (x, y) \mapsto x \circ y$$

is a hyperoperation. For instance,

$$\mathsf{Stable} \circ \mathsf{Improved} = \{\mathsf{Improved}, \mathsf{Stable}\},$$

$$\mathsf{MildSideEffect} \circ \mathsf{Improved} = \{\mathsf{Improved}, \mathsf{MildSideEffect}, \mathsf{SevereSideEffect}\}.$$

Hence $(S, \circ)$ is a hyperstructure capturing multivalued clinical outcomes of combining treatments. For reference, a schematic overview of this concrete example is shown in Fig. 2.1.

Example 2.1.7 (Route planning with uncertain transfers as a hyperstructure)**.** *Interpretation.* Let S be a set of travel states in an urban transport network, such as

$$S = \{\mathsf{OnTime},\ \mathsf{MinorDelay},\ \mathsf{MissedConnection},\ \mathsf{Reroute},\ \mathsf{Arrived}\}.$$

When a commuter combines two legs of a trip, the resulting state is not deterministic because of traffic, platform congestion, and schedule variability.

Hyperoperation. For $x, y \in S$, define $x \circ y \subseteq S$ as the set of possible travel states after completing a leg in state x and then attempting a transfer/next leg in state y. Then $\circ : S \times S \to \mathcal{P}(S)$ is a hyperoperation. For example,

$$\mathsf{OnTime} \circ \mathsf{OnTime} = \{\mathsf{Arrived}\}, \qquad \mathsf{MinorDelay} \circ \mathsf{OnTime} = \{\mathsf{Arrived}, \mathsf{MinorDelay}, \mathsf{MissedConnection}\},$$

$$\mathsf{MissedConnection} \circ \mathsf{OnTime} = \{\mathsf{Reroute}, \mathsf{Arrived}\}.$$

Thus $(S, \circ)$ forms a hyperstructure modeling multi-outcome composition of trip segments under uncertainty.

A *superhyperstructure* refines this idea by iterating the powerset construction. Instead of working only with $\mathcal{P}(S)$, one allows operations whose inputs and outputs live in higher iterated powersets $\mathcal{P}^n(S)$, so that operations may act on nested collections. This provides a natural framework for multi–layer hierarchical interactions [30, 33, 34]. Typical examples include SuperHyperGraphs [15, 16, 35] and SuperHyperUncertain Sets [19].

Definition 2.1.8 (Iterated powersets)**.** (cf. [1]) Let S be a nonempty set. Define recursively, for $n \geq 0$,

$$\mathcal{P}^0(S) = S, \qquad \mathcal{P}^{n+1}(S) = \mathcal{P}\big(\mathcal{P}^n(S)\big).$$

Thus $\mathcal{P}^1(S) = \mathcal{P}(S)$, $\mathcal{P}^2(S) = \mathcal{P}(\mathcal{P}(S))$, etc.

Define likewise the *nonempty* iterated powersets by

$$\mathcal{P}_*^0(S) = S, \qquad \mathcal{P}_*^{n+1}(S) = \mathcal{P}_*\big(\mathcal{P}_*^n(S)\big) = \mathcal{P}\big(\mathcal{P}_*^n(S)\big) \setminus \{\emptyset\} \quad (n \geq 0).$$

In particular, $\mathcal{P}_*^1(S) = \mathcal{P}_*(S)$.

We will frequently use the standard embedding $\eta_n : S \hookrightarrow \mathcal{P}^n(S)$ to view classical operations as special cases of superhyperoperations.

Definition 2.1.9 (Standard (unit) embedding into iterated powersets)**.** Let S be a nonempty set and $n \in \mathbb{N}_0$. Define the *n-fold unit embedding*

$$\eta_n : S \longrightarrow \mathcal{P}^n(S)$$

recursively by

$$\eta_0(x) := x, \qquad \eta_{n+1}(x) := \{\eta_n(x)\} \quad (n \geq 0).$$

We call η_n the *standard embedding* (or *unit embedding*) of S into $\mathcal{P}^n(S)$.

Lemma 2.1.10 (Injectivity of the unit embedding)**.** *For every $n \in \mathbb{N}_0$, the map $\eta_n : S \to \mathcal{P}^n(S)$ is injective.*

Proof. We argue by induction on n. For $n = 0$, $\eta_0 = \mathrm{id}_S$ is injective. Assume η_n is injective. If $\eta_{n+1}(x) = \eta_{n+1}(y)$, then $\{\eta_n(x)\} = \{\eta_n(y)\}$, hence $\eta_n(x) = \eta_n(y)$, and by the induction hypothesis $x = y$. Thus η_{n+1} is injective. □

Definition 2.1.11 (Canonical lifting of a classical operation to level n)**.** Let $k \in \mathbb{N}$ and let $f : S^k \to S$ be a (classical) k-ary operation. Define its *canonical level-n lift*

$$f^{\langle n \rangle} : \big(\mathcal{P}^n(S)\big)^k \longrightarrow \mathcal{P}^n(S)$$

recursively by

$$f^{\langle 0 \rangle} := f,$$

and for $n \geq 0$,

$$f^{\langle n+1 \rangle}(A_1, \ldots, A_k) := \Big\{ f^{\langle n \rangle}(a_1, \ldots, a_k) \;\Big|\; a_i \in A_i \ (1 \leq i \leq k) \Big\},$$

where $A_i \in \mathcal{P}^{n+1}(S) = \mathcal{P}(\mathcal{P}^n(S))$.

Proposition 2.1.12 (Recovery of classical operations as a special case)**.** *Let $f : S^k \to S$ be a classical k-ary operation and let $n \in \mathbb{N}_0$. Then, for all $x_1, \ldots, x_k \in S$,*

$$f^{\langle n \rangle}\big(\eta_n(x_1), \ldots, \eta_n(x_k)\big) = \eta_n\big(f(x_1, \ldots, x_k)\big).$$

In particular, the classical operation f is recovered as the restriction of the super-level operation $f^{\langle n \rangle}$ to the embedded copy $\eta_n(S) \subseteq \mathcal{P}^n(S)$:

$$\big(\eta_n(S),\ f^{\langle n \rangle}{\restriction}_{\eta_n(S)^k}\big) \cong (S, f),$$

via the bijection η_n.

Proof. We prove the displayed identity by induction on n.

Base $n = 0$. We have $f^{\langle 0 \rangle}(\eta_0(x_1), \ldots, \eta_0(x_k)) = f(x_1, \ldots, x_k) = \eta_0(f(x_1, \ldots, x_k))$.

Step $n \to n + 1$. Using the recursive definitions,

$$f^{\langle n+1 \rangle}\big(\eta_{n+1}(x_1), \ldots, \eta_{n+1}(x_k)\big) = f^{\langle n+1 \rangle}\big(\{\eta_n(x_1)\}, \ldots, \{\eta_n(x_k)\}\big).$$

By Definition 2.1.11, the right-hand side equals the singleton set

$$\Big\{ f^{\langle n \rangle}(\eta_n(x_1), \ldots, \eta_n(x_k)) \Big\}.$$

By the induction hypothesis, $f^{\langle n \rangle}(\eta_n(x_1), \ldots, \eta_n(x_k)) = \eta_n(f(x_1, \ldots, x_k))$, hence the set above is

$$\{\eta_n(f(x_1, \ldots, x_k))\} = \eta_{n+1}(f(x_1, \ldots, x_k)).$$

This proves the identity for $n + 1$.

Finally, since η_n is injective (Lemma 2.1.10), it yields an isomorphism between (S, f) and the restriction of $(\mathcal{P}^n(S), f^{\langle n \rangle})$ to $\eta_n(S)$. □

Definition 2.1.13 ((m,n)-superhyperoperation)**.** [1] Let S be a nonempty set, and let $m, n \geq 0$ and $k \geq 1$ be integers. A *k-ary (m,n)-superhyperoperation* on S is a map

$$\star : \underbrace{\mathcal{P}^m(S) \times \cdots \times \mathcal{P}^m(S)}_{k \text{ factors}} \longrightarrow \mathcal{P}^n(S).$$

.

Definition 2.1.14 (n-superhyperstructure)**.** (cf. [1, 33, 36]) Fix $n \geq 0$ and $k \geq 1$. An *n-superhyperstructure (of arity k)* on a base set S is a pair

$$\mathcal{SH}^{(n)} = (\mathcal{P}^n(S), \star),$$

where $\star : (\mathcal{P}^n(S))^k \to \mathcal{P}^n(S)$ is a k-ary operation at level n (often taken to be classical-type by replacing $\mathcal{P}^n$ with $\mathcal{P}^n_*$, if desired). For $n = 0$ this is an ordinary k-ary algebraic structure on S.

Example 2.1.15 (Incident-response playbook libraries as a 2-superhyperstructure)**.** *Base set.* Let S be a set of atomic response actions in a security operations center, e.g.,

$$S = \{\text{IsolateHost},\ \text{ResetCreds},\ \text{BlockIP},\ \text{Patch},\ \text{NotifyUsers}\}.$$

An element of $\mathcal{P}^2(S) = \mathcal{P}(\mathcal{P}(S))$ is a *library* of action-bundles (each bundle is a subset of S).

Operation. Fix $k = 2$ and define a binary operation

$$\star : \mathcal{P}^2(S) \times \mathcal{P}^2(S) \longrightarrow \mathcal{P}^2(S)$$

by

$$A \star B \ := \ \{\, U \cup V \ \mid \ U \in A,\ V \in B \,\}.$$

Then $U \cup V \subseteq S$, hence $A \star B \subseteq \mathcal{P}(S)$, so $A \star B \in \mathcal{P}^2(S)$. Thus $\mathcal{SH}^{(2)} = (\mathcal{P}^2(S), \star)$ is a 2-superhyperstructure: combining two playbook-libraries yields the library of all bundle-unions.

Example 2.1.16 (Cross-team project portfolio merging as a 3-superhyperstructure)**.** *Base set.* Let S be a set of atomic tasks, e.g.,

$$S = \{\text{Design},\ \text{Implement},\ \text{Test},\ \text{Deploy},\ \text{Document}\}.$$

A level-1 object is a task-bundle $B \subseteq S$. A level-2 object is a *portfolio* $P \in \mathcal{P}^2(S)$, i.e. a set of bundles. A level-3 object is a *collection of portfolios* $\mathfrak{P} \in \mathcal{P}^3(S) = \mathcal{P}(\mathcal{P}^2(S))$.

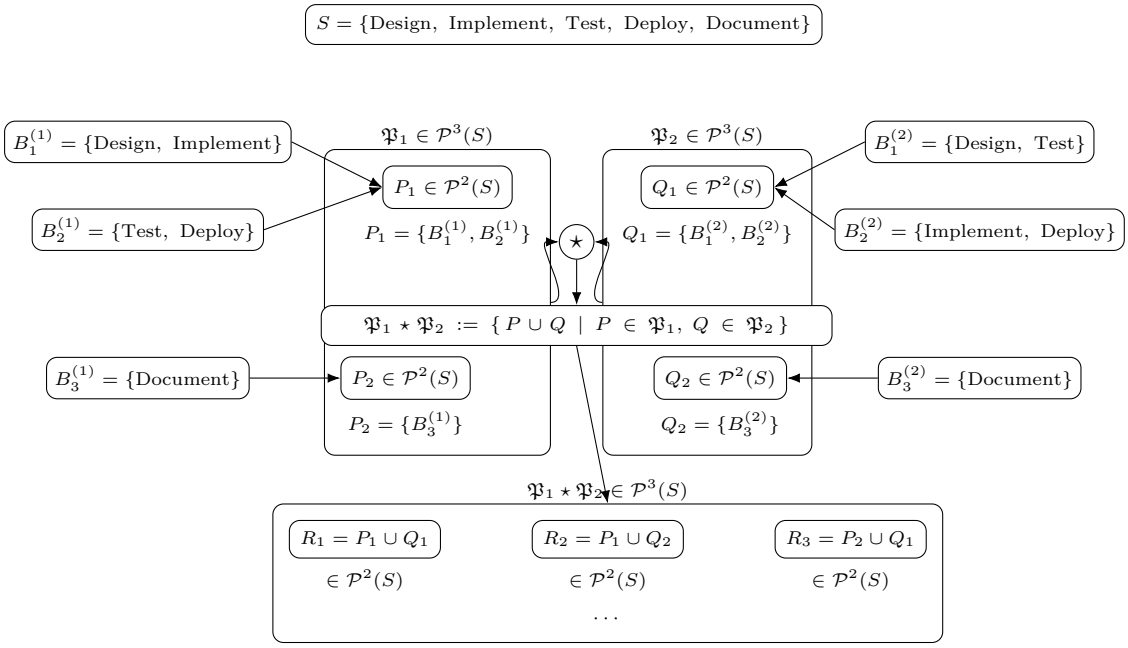
Operation. Fix $k = 2$ and define

$$\star : \mathcal{P}^3(S) \times \mathcal{P}^3(S) \longrightarrow \mathcal{P}^3(S)$$

by portfolio-union at the leaves:

$$\mathfrak{P}_1 \star \mathfrak{P}_2 \ := \ \{\, P_1 \cup P_2 \ \mid \ P_1 \in \mathfrak{P}_1,\ P_2 \in \mathfrak{P}_2 \,\}.$$

Since each $P_1 \cup P_2 \subseteq \mathcal{P}(S)$, we have $P_1 \cup P_2 \in \mathcal{P}^2(S)$, hence $\mathfrak{P}_1 \star \mathfrak{P}_2 \subseteq \mathcal{P}^2(S)$, i.e. it is an element of $\mathcal{P}^3(S)$. Therefore $\mathcal{SH}^{(3)} = (\mathcal{P}^3(S), \star)$ is a 3-superhyperstructure.



Levels: Level-1 bundle $B \subseteq S$; Level-2 portfolio $P \in \mathcal{P}^2(S)$ (set of bundles); Level-3 collection $\mathfrak{P} \in \mathcal{P}^3(S)$ (set of portfolios).

Figure 2.2: Overview of cross-team portfolio merging as a 3-superhyperstructure $\mathcal{SH}^{(3)} = (\mathcal{P}^3(S), \star)$ with $\star$ defined by portfolio-union at the leaves.

Definition 2.1.17 ((m,n)-superhyperstructure)**.** Let $S \neq \varnothing$, and let $m, n \in \mathbb{N}_0$, $k \geq 1$. An (m, n)*-superhyperstructure of arity* k *over* S is a pair

$$\mathrm{SH}^{(m,n)} = (\mathcal{P}^m(S), \star),$$

where $\star : (\mathcal{P}^m(S))^k \to \mathcal{P}^n(S)$.

Example 2.1.18 (Team-skill aggregation as a $(1, 1)$-superhyperstructure)**.** *Base set.* Let S be a (finite or infinite) set of skills, certifications, or capabilities (e.g., $\{\mathsf{Kubernetes}, \mathsf{Python}, \mathsf{SOC2}, \mathsf{PCI\text{-}DSS}, \dots\}$).

Underlying domain. A level-1 element $A \in \mathcal{P}(S)$ represents the skill set of a team member or a team.

Operation type. Take $m = n = 1$ and $k = 2$. Define

$$\star : \mathcal{P}(S) \times \mathcal{P}(S) \longrightarrow \mathcal{P}(S), \qquad A \star B := A \cup B.$$

Well-definedness. For any $A, B \subseteq S$, we have $A \cup B \subseteq S$, hence $A \star B \in \mathcal{P}(S) = \mathcal{P}^1(S)$. Therefore $\mathrm{SH}^{(1,1)} = (\mathcal{P}(S), \star)$ is a $(1, 1)$-superhyperstructure of arity 2.

Interpretation. If A and B are the skill sets of two teams (or two candidates), then $A \star B$ is the combined capability set available after merging/teaming up.

Example 2.1.19 (Cross-department agreement pool as a $(2,1)$-superhyperstructure)**.** *Base set.* Let S be a set of possible actions or controls (e.g., incident-response steps, mitigations, policy controls, deployment tasks).

Underlying domain. A level-2 element $X \in \mathcal{P}^2(S) = \mathcal{P}(\mathcal{P}(S))$ is a *set of candidate bundles*

$$X = \{A_1, A_2, \dots\}, \qquad A_i \subseteq S.$$

Think of X as a library of feasible plans proposed by one department, team, or stakeholder group.

Operation type. Take $m = 2$, $n = 1$, and $k = 2$. Define

$$\star : \mathcal{P}^2(S) \times \mathcal{P}^2(S) \longrightarrow \mathcal{P}(S)$$

by

$$X \star Y \;:=\; \bigcup_{A \in X} \bigcup_{B \in Y} (A \cap B).$$

Well-definedness. For every $A \in X$ and $B \in Y$, we have $A \cap B \subseteq S$. Hence the union of such intersections is again a subset of S, so $X \star Y \in \mathcal{P}(S) = \mathcal{P}^1(S)$. Therefore $\mathrm{SH}^{(2,1)} = (\mathcal{P}^2(S), \star)$ is a $(2,1)$-superhyperstructure of arity 2.

Interpretation. The inputs X and Y are two collections of feasible plans. The output $X \star Y$ collects all actions that appear in at least one *shared subplan* $A \cap B$ between some $A \in X$ and some $B \in Y$. Thus $X \star Y$ can be read as an "agreement pool": items that two groups can jointly endorse within at least one compatible pair of proposals.

Chapter 3

HyperStructure and SuperHyperStructure in Algebra

In this chapter, we investigate hyperstructures and superhyperstructures in algebra and related concepts. For reference, Table 3.1 lists classical algebraic concepts together with their hyper and superhyper counterparts.

3.1 HyperAlgebra and SuperHyperAlgebra

Algebra studies symbols, operations, and rules for manipulating expressions and solving equations [37–39]. Related concepts such as fuzzy algebra [40,41], Linear algebra, Elementary algebra, and neutro-algebra [42,43] are also known. In many applications, however, an "operation" naturally returns a *set* of possible outputs rather than a single output. This motivates *hyperalgebras*, whose basic operations are hyperoperations [44–47]. As related concepts of hyperalgebras, fuzzy hyperalgebras [48, 49] and neutrosophic hyperalgebras [50, 51] are also known. A further iterated generalization, called a *superhyperalgebra*, allows operations to take values in higher iterated powersets [52–55]. Throughout this section we use the powerset notation $\mathcal{P}_*(H)$ and the iterated nonempty powersets $\mathcal{P}_*^n(H)$ introduced earlier.

Definition 3.1.1 (Hyperalgebra)**.** [36] Let H be a nonempty set, and let Σ be an algebraic signature (a set of operation symbols) equipped with an arity map $\mathrm{ar} : \Sigma \to \mathbb{N}_{\geq 1}$. A *hyperalgebra* of signature Σ is a structure

$$\mathbf{H} = \big(H,\ (\sigma^{\mathbf{H}})_{\sigma\in\Sigma}\big),$$

where each basic operation is interpreted as a hyperoperation

$$\sigma^{\mathbf{H}} : H^{\mathrm{ar}(\sigma)} \longrightarrow \mathcal{P}_*(H) \qquad (\sigma \in \Sigma).$$

Equivalently, for every $(x_1, \ldots, x_{\mathrm{ar}(\sigma)}) \in H^{\mathrm{ar}(\sigma)}$, the value $\sigma^{\mathbf{H}}(x_1, \ldots, x_{\mathrm{ar}(\sigma)})$ is a *nonempty* subset of H.

If $\mathcal{A}$ is a chosen set of axioms (e.g., identities, inclusions, or other constraints formulated for hyperoperations), then we write $\mathbf{H} = (H, (\sigma^{\mathbf{H}})_{\sigma\in\Sigma}, \mathcal{A})$ and say that $\mathbf{H}$ is a hyperalgebra *satisfying* $\mathcal{A}$.

Classical Concept	**HyperConcept**	**SuperHyperConcept**
Algebra	HyperAlgebra	SuperHyperAlgebra
Group	HyperGroup	SuperHyperGroup
Topology	HyperTopology	SuperHyperTopology
Lattice	HyperLattice	SuperHyperLattice
Field	HyperField	SuperHyperField
Ring	HyperRing	SuperHyperRing
Vector space	HyperVector	SuperHyperVector
Module	HyperModule	SuperHyperModule
Matrix / Matrices	HyperMatrix (HyperMatrices)	SuperHyperMatrix (SuperHyperMatrices)
Category	HyperCategory	SuperHyperCategory
Magma	Hypermagma	SuperHypermagma
Monoid	HyperMonoid	SuperHyperMonoid
Ideal	HyperIdeal	SuperHyperIdeal

Table 3.1: Correspondence of classical concepts with their hyper and superhyper counterparts.

Definition 3.1.2 (Superhyperalgebra)**.** [36] Fix an integer $n \geq 1$. Let H be a nonempty set and let Σ be a signature with arity map $\mathrm{ar} : \Sigma \to \mathbb{N}_{\geq 1}$. A *(level-$n$) superhyperalgebra* of signature Σ is a structure

$$\mathbf{SH}^{(n)} = \big(H,\ (\sigma^{\mathbf{SH}})_{\sigma\in\Sigma}\big),$$

where each basic operation is interpreted as a map into the n-th iterated nonempty powerset:

$$\sigma^{\mathbf{SH}} : H^{\mathrm{ar}(\sigma)} \longrightarrow \mathcal{P}^n_*(H) \qquad (\sigma \in \Sigma).$$

Thus, for every input tuple, $\sigma^{\mathbf{SH}}$ returns a *nested* nonempty collection, i.e., an element of $\mathcal{P}^n_*(H)$.

If $\mathcal{A}$ is a chosen set of axioms adapted to such outputs, we write $\mathbf{SH}^{(n)} = (H, (\sigma^{\mathbf{SH}})_{\sigma\in\Sigma}, \mathcal{A})$. The special case $n = 1$ recovers hyperalgebras. Allowing the empty set at the output level amounts to replacing $\mathcal{P}^n_*(H)$ by $\mathcal{P}^n(H)$; this convention is sometimes termed *neutrosophic-type* in applications.

Example 3.1.3 (A simple hyperalgebra and its level-2 superhyperalgebra lifting)**.** Let $H = \{0, 1\}$ and take the signature $\Sigma = \{\oplus\}$ with $\mathrm{ar}(\oplus) = 2$.

Hyperalgebra. Define a binary hyperoperation $\oplus^{\mathbf{H}} : H \times H \to \mathcal{P}_*(H)$ by

$$0 \oplus^{\mathbf{H}} 0 = \{0\}, \qquad 0 \oplus^{\mathbf{H}} 1 = 1 \oplus^{\mathbf{H}} 0 = \{0, 1\}, \qquad 1 \oplus^{\mathbf{H}} 1 = \{1\}.$$

Then $\mathbf{H} = (H, (\oplus^{\mathbf{H}}))$ is a hyperalgebra of signature Σ. For instance, the "sum" of 0 and 1 is ambiguous and may yield either 0 or 1.

Level-2 superhyperalgebra (encoding uncertainty hierarchically). Define $\oplus^{\mathbf{SH}} : H \times H \to \mathcal{P}^2_*(H) = \mathcal{P}_*(\mathcal{P}_*(H))$ by

$$a \oplus^{\mathbf{SH}} b \ := \ \big\{\ \{a\},\ \{b\},\ a \oplus^{\mathbf{H}} b\ \big\} \qquad (a, b \in H).$$

Each output is a nonempty *family* of nonempty subsets of H, hence lies in $\mathcal{P}^2_*(H)$. Concretely,

$$0 \oplus^{\mathbf{SH}} 1 = \big\{\{0\}, \{1\}, \{0, 1\}\big\}, \qquad 1 \oplus^{\mathbf{SH}} 1 = \big\{\{1\}\big\}.$$

Thus $\mathbf{SH}^{(2)} = (H, (\oplus^{\mathbf{SH}}))$ is a level-2 superhyperalgebra whose basic operation returns a nested collection: two "certain" singleton outcomes plus the original hyper-sum.

3.2 HyperGroup and SuperHyperGroup

Hypergroups generalize groups by replacing the ordinary binary operation with a *set-valued* product (a hyperoperation) [56–58]. Superhypergroups refine this idea by allowing the product of two elements to be a *nested* set, i.e., an element of an iterated powerset level [59].

Definition 3.2.1 (Hypergroup)**.** (cf. [56–58]) Let H be a nonempty set and let $\star : H \times H \to \mathcal{P}_*(H)$ be a (total) binary hyperoperation. Let the induced product on subsets be the canonical one (as fixed earlier), so that for $A, B \subseteq H$ one has $A \star B \subseteq H$.

A *hypergroup* is a triple $(H, \star, e)$ such that:

1. **(Associativity)** For all $x, y, z \in H$,
$$(x \star y) \star z \;=\; x \star (y \star z) \qquad \text{as subsets of } H.$$

2. **(Identity)** There exists $e \in H$ such that for all $x \in H$,
$$e \star x = \{x\} \qquad \text{and} \qquad x \star e = \{x\}.$$

3. **(Inverses)** For each $x \in H$ there exists an element $x^{-1} \in H$ such that
$$e \in x \star x^{-1} \qquad \text{and} \qquad e \in x^{-1} \star x.$$

(Some authors weaken the identity axiom to $x \in e \star x \cap x \star e$, and formulate inverses via a reversibility axiom; the above is a standard group-like variant.)

Example 3.2.2 (A nontrivial finite hypergroup: the Krasner hypergroup)**.** Let $H = \{0, 1\}$ and define a total hyperoperation $\star : H \times H \to \mathcal{P}_*(H)$ by
$$0 \star 0 = \{0\}, \qquad 0 \star 1 = 1 \star 0 = \{1\}, \qquad 1 \star 1 = \{0, 1\}.$$
Let $e := 0$. Then:

- **Identity:** $e \star x = \{x\} = x \star e$ for $x \in H$ is immediate from the table.
- **Inverses:** $0^{-1} = 0$ since $0 \in 0 \star 0$, and $1^{-1} = 1$ since $0 \in 1 \star 1 = \{0, 1\}$.
- **Associativity:** using the canonical subset extension, it suffices to check the nontrivial case involving $1 \star 1$:
$$(1 \star 1) \star 1 = \{0, 1\} \star 1 = (0 \star 1) \cup (1 \star 1) = \{1\} \cup \{0, 1\} = \{0, 1\},$$
$$1 \star (1 \star 1) = 1 \star \{0, 1\} = (1 \star 0) \cup (1 \star 1) = \{1\} \cup \{0, 1\} = \{0, 1\}.$$
All remaining cases are singleton-valued and therefore associative as in an ordinary semigroup.

Hence $(H, \star, e)$ is a hypergroup in the sense of Definition 3.2.1.

Definition 3.2.3 (Superhypergroup)**.** (cf. [2, 60]) Fix an integer $n \geq 1$. Let H be a nonempty set and let $\circ : H \times H \to \mathcal{P}_*^n(H)$ be a (total) level-n superhyperoperation.

Define the *nested singleton embedding* $\iota_n : H \to \mathcal{P}_*^n(H)$ by

$$\iota_0(x) := x, \qquad \iota_{k+1}(x) := \{\iota_k(x)\} \quad (k \geq 0).$$

Define also the *flattening* maps $\mathrm{Flat}_k : \mathcal{P}_*^k(H) \to \mathcal{P}_*(H)$ recursively by

$$\mathrm{Flat}_0(x) := \{x\}, \qquad \mathrm{Flat}_{k+1}(X) := \bigcup_{A \in X} \mathrm{Flat}_k(A) \quad (k \geq 0),$$

so that $\mathrm{Flat}_n(X) \subseteq H$ is the underlying (nonempty) set of elements appearing in the nested object X.

Extend $\circ$ to act on level-n arguments by

$$X \circ Y \ := \bigcup_{u \in \mathrm{Flat}_n(X)} \ \bigcup_{v \in \mathrm{Flat}_n(Y)} (u \circ v), \qquad X, Y \in \mathcal{P}_*^n(H),$$

and identify each element $x \in H$ with $\iota_n(x) \in \mathcal{P}_*^n(H)$.

A *superhypergroup (of level n)* is a structure $(H, \circ, e)$ such that:

1. **(Strong superhyperassociativity)** For all $x, y, z \in H$,
$$x \circ (y \circ z) \ = \ (x \circ y) \circ z \qquad \text{in } \mathcal{P}_*^n(H),$$
where the left- and right-hand sides use the above extension and the identification $x \simeq \iota_n(x)$.

2. **(Identity)** There exists $e \in H$ such that for all $x \in H$,
$$e \circ x \ = \ \iota_n(x) \qquad \text{and} \qquad x \circ e \ = \ \iota_n(x).$$

3. **(Inverses)** For each $x \in H$ there exists $x^{-1} \in H$ such that
$$e \in \mathrm{Flat}_n(x \circ x^{-1}) \qquad \text{and} \qquad e \in \mathrm{Flat}_n(x^{-1} \circ x).$$
(Equivalently, $\iota_{n-1}(e) \in x \circ x^{-1} \cap x^{-1} \circ x$, since elements of $x \circ y \in \mathcal{P}_*^n(H)$ live at level $n-1$.)

For $n = 1$, this reduces to the group-like hypergroup axioms in Definition 3.2.1.

Example 3.2.4 (A level-2 superhypergroup induced by the Krasner hypergroup)**.** Let $(H, \star, e)$ be the hypergroup from Example 3.2.2, with $H = \{0, 1\}$ and $e = 0$. Fix $n = 2$. Define a level-2 superhyperoperation $\circ : H \times H \to \mathcal{P}_*^2(H)$ by the *singleton-lift* of $\star$:

$$x \circ y \ := \ \big\{\{u\} \mid u \in x \star y\big\} \qquad (x, y \in H).$$

Concretely,

$$0 \circ 0 = \big\{\{0\}\big\}, \quad 0 \circ 1 = 1 \circ 0 = \big\{\{1\}\big\}, \quad 1 \circ 1 = \big\{\{0\}, \{1\}\big\},$$

which indeed lie in $\mathcal{P}_*^2(H) = \mathcal{P}_*(\mathcal{P}_*(H))$.

We verify the superhypergroup axioms of Definition 3.2.3:

- **Identity:** since $e \star x = \{x\}$, we have

$$e \circ x = \big\{\{x\}\big\} = \iota_2(x), \qquad x \circ e = \big\{\{x\}\big\} = \iota_2(x),$$

so e is an identity in the required sense.

- **Inverses:** take x^{-1} as in Example 3.2.2. For $x = 1$, $1^{-1} = 1$ and

$$1 \circ 1 = \big\{\{0\}, \{1\}\big\}, \qquad \mathrm{Flat}_2(1 \circ 1) = \{0, 1\} \ni e.$$

For $x = 0$, $0^{-1} = 0$ and $\mathrm{Flat}_2(0 \circ 0) = \{0\} \ni e$.

- **Strong superhyperassociativity:** using the extension rule in Definition 3.2.3, note that $\mathrm{Flat}_2(y \circ z) = y \star z$. Hence

$$x \circ (y \circ z) = \bigcup_{v \in \mathrm{Flat}_2(y \circ z)} (x \circ v) = \bigcup_{v \in (y \star z)} \big\{\{w\} \mid w \in x \star v\big\} = \big\{\{w\} \mid w \in x \star (y \star z)\big\},$$

and similarly

$$(x \circ y) \circ z = \big\{\{w\} \mid w \in (x \star y) \star z\big\}.$$

Since $\star$ is associative as a hyperoperation, $x \star (y \star z) = (x \star y) \star z$, so the two displayed sets are equal in $\mathcal{P}^2_*(H)$.

Therefore $(H, \circ, e)$ is a level-2 superhypergroup.

3.3 HyperTopology and SuperHyperTopology

A *topology* on a set X is a family of subsets of X that contains $\emptyset$ and X, is closed under arbitrary unions, and is closed under finite intersections [61–63]. Related concepts such as fuzzy topology [64–66] and neutrotopology [67,68] are also known. A *hypertopology* is simply a topology whose underlying set is the powerset $\mathcal{P}(H)$; hence its "points" are subsets of H, and its "open sets" are families of subsets of H [69, 70]. More generally, a *superhypertopology of level n* is a topology on the iterated powerset level $\mathcal{P}^n(H)$ (or, depending on convention, on its nonempty part), so that one may treat nested collections as points [71–73].

Definition 3.3.1 (Hypertopology)**.** (cf. [69, 70]) Let H be a nonempty set. A *hypertopology* on H is a topology $\tau_H \subseteq \mathcal{P}(\mathcal{P}(H))$ on the underlying set $\mathcal{P}(H)$; that is, τ_H is a family of subsets of $\mathcal{P}(H)$ satisfying:

1. $\emptyset \in \tau_H$ and $\mathcal{P}(H) \in \tau_H$;

2. if $\{V_\beta\}_{\beta \in J} \subseteq \tau_H$ is any (possibly infinite) family, then $\bigcup_{\beta \in J} V_\beta \in \tau_H$;

3. if $V_1, \ldots, V_m \in \tau_H$ with $m \geq 1$, then $\bigcap_{j=1}^{m} V_j \in \tau_H$.

The pair $(\mathcal{P}(H), \tau_H)$ is called a *hypertopological space*.

Definition 3.3.2 (Superhypertopology)**.** (cf. [71,73]) Let H be a nonempty set and fix an integer $n \geq 1$. A *superhypertopology of level n* on H is a topology $\tau_{SH} \subseteq \mathcal{P}(\mathcal{P}^n(H))$ on the underlying set $\mathcal{P}^n(H)$. Equivalently, τ_{SH} is a family of subsets of $\mathcal{P}^n(H)$ such that:

1. $\emptyset \in \tau_{SH}$ and $\mathcal{P}^n(H) \in \tau_{SH}$;

2. if $\{W_\gamma\}_{\gamma \in K} \subseteq \tau_{SH}$ is any family, then $\bigcup_{\gamma \in K} W_\gamma \in \tau_{SH}$;

3. if $W_1, \ldots, W_p \in \tau_{SH}$ with $p \geq 1$, then $\bigcap_{i=1}^{p} W_i \in \tau_{SH}$.

The pair $(\mathcal{P}^n(H), \tau_{SH})$ is called a *superhypertopological space of level n*.

Remark. If one wishes to *exclude* empty sets at each level, one may replace $\mathcal{P}^n(H)$ by the nonempty iterated powerset $\mathcal{P}^n_*(H)$ defined earlier, and define a topology on $\mathcal{P}^n_*(H)$ instead.

Example 3.3.3 (A level-2 superhypertopology induced from a finite base)**.** Let $H = \{a, b\}$ and consider the level $n = 2$ carrier

$$\mathcal{P}^2(H) = \mathcal{P}(\mathcal{P}(H)).$$

Define the *nonempty-membership selector*

$$\mathsf{NE} := \{\, \mathcal{A} \in \mathcal{P}^2(H) \;\mid\; \emptyset \notin \mathcal{A} \,\} \;\subseteq\; \mathcal{P}^2(H).$$

Now define

$$\tau_{SH} = \{\emptyset,\ \mathcal{P}^2(H),\ \mathsf{NE}\} \;\subseteq\; \mathcal{P}(\mathcal{P}^2(H)).$$

We check the topology axioms on the underlying set $\mathcal{P}^2(H)$:

- $\emptyset \in \tau_{SH}$ and $\mathcal{P}^2(H) \in \tau_{SH}$ are explicit.

- **Arbitrary unions:** any union of members of τ_{SH} is one of $\emptyset, \mathsf{NE}, \mathcal{P}^2(H)$, hence belongs to τ_{SH}.

- **Finite intersections:** the only nontrivial intersection is

  $$\mathsf{NE} \cap \mathcal{P}^2(H) = \mathsf{NE},$$

  and intersections with $\emptyset$ are $\emptyset$. Hence τ_{SH} is closed under finite intersections.

Therefore τ_{SH} is a topology on $\mathcal{P}^2(H)$, and $(\mathcal{P}^2(H), \tau_{SH})$ is a level-2 superhypertopological space in the sense of Definition 3.3.2.

3.4 HyperLattice and SuperHyperLattice

HyperLattice is a lattice-like structure where one operation is ordinary meet/join, while the other is a set-valued hyperoperation satisfying absorption [74, 75]. Related concepts such as fuzzy hyperlattices [74,76–78] are also known. SuperHyperLattice iterates HyperLattice to higher powerset levels, so operations act on nested subsets, preserving lifted associativity and absorption [79].

Definition 3.4.1 (Commutative hypergroup (reproductive form))**.** [80,81] Let L be a nonempty set and let $\circ : L \times L \to \mathcal{P}_*(L)$ be a (total) binary hyperoperation. Using the canonical subset extension of $\circ$, we write for $A, B \subseteq L$

$$A \circ B := \bigcup_{a\in A} \bigcup_{b\in B} (a \circ b) \subseteq L, \qquad a \circ A := \{a\} \circ A, \quad A \circ a := A \circ \{a\}.$$

The pair $(L, \circ)$ is called a *commutative hypergroup in reproductive form* if:

1. **(Commutativity)** $a \circ b = b \circ a$ for all $a, b \in L$.

2. **(Associativity)** $(a \circ b) \circ c = a \circ (b \circ c)$ for all $a, b, c \in L$, where both sides are computed via the subset extension above.

3. **(Reproductivity)** $a \circ L = L \circ a = L$ for all $a \in L$.

Example 3.4.2 (A commutative reproductive hypergroup on a two-element set)**.** Let $L = \{0, 1\}$ and define $\circ : L \times L \to \mathcal{P}_*(L)$ by

$$0 \circ 0 = \{0\}, \qquad 0 \circ 1 = 1 \circ 0 = \{0, 1\}, \qquad 1 \circ 1 = \{1\}.$$

Commutativity is immediate from the symmetric table.

To check *associativity* in the sense of Definition 3.4.1, we use the canonical subset extension $A \circ B = \bigcup_{a\in A, b\in B}(a \circ b)$. The only nontrivial case is when an intermediate product equals $\{0, 1\}$. For example,

$$(0 \circ 1) \circ 1 = \{0, 1\} \circ 1 = (0 \circ 1) \cup (1 \circ 1) = \{0, 1\} \cup \{1\} = \{0, 1\},$$

while

$$0 \circ (1 \circ 1) = 0 \circ \{1\} = 0 \circ 1 = \{0, 1\}.$$

All remaining triples $(a, b, c) \in L^3$ either reduce to singleton-valued steps or are analogous, and one verifies that $(a \circ b) \circ c = a \circ (b \circ c)$ as subsets of L.

Finally, *reproductivity* holds: for each $a \in L$,

$$a \circ L = a \circ \{0, 1\} = (a \circ 0) \cup (a \circ 1) = \{0, 1\} = L,$$

and similarly $L \circ a = L$. Hence $(L, \circ)$ is a commutative hypergroup in reproductive form.

Definition 3.4.3 (Hyperlattice (join-hyperlattice))**.** [82, 83] A *(join-)hyperlattice* is a triple $(L, \wedge, \circ)$ such that:

1. L is a nonempty set;

2. $\wedge : L \times L \to L$ makes $(L, \wedge)$ a commutative idempotent semigroup (equivalently, a meet-semilattice), i.e.

$$a \wedge b = b \wedge a, \qquad (a \wedge b) \wedge c = a \wedge (b \wedge c), \qquad a \wedge a = a;$$

3. $\circ : L \times L \to \mathcal{P}_*(L)$ makes $(L, \circ)$ a commutative hypergroup in the sense of Definition 3.4.1;

4. **(Absorption-type laws)** for all $a, b \in L$,

$$a \in a \circ (a \wedge b) \qquad \text{and} \qquad a \in a \wedge (a \circ b),$$

where for $A \subseteq L$ we use the standard notation

$$a \wedge A \;:=\; \{\, a \wedge x \mid x \in A \,\} \subseteq L, \quad \text{so that} \quad a \wedge (a \circ b) = \{\, a \wedge x \mid x \in a \circ b \,\}.$$

(Depending on the intended subclass, one may additionally impose distributivity-type conditions, e.g. $a \wedge (b \circ c) \subseteq (a \wedge b) \circ (a \wedge c)$, etc.)

Example 3.4.4 (A hyperlattice built from min and a reproductive hypergroup)**.** Let $L = \{0, 1\}$ and keep the hyperoperation $\circ$ from Example 3.4.2. Define a meet operation $\wedge : L \times L \to L$ by

$$a \wedge b := \min\{a, b\}.$$

Then $(L, \wedge)$ is a commutative idempotent semigroup: min is commutative and associative and satisfies $a \wedge a = a$.

We now verify the absorption-type laws of Definition 3.4.3. For any $a, b \in L$, compute $a \wedge b \in L$ and check:

$$a \in a \circ (a \wedge b).$$

If $a = 0$, then $a \wedge b = 0$ and $0 \circ 0 = \{0\} \ni 0$. If $a = 1$, then $a \wedge b = b$; when $b = 0$, $1 \circ 0 = \{0, 1\} \ni 1$, and when $b = 1$, $1 \circ 1 = \{1\} \ni 1$.

Next, verify

$$a \in a \wedge (a \circ b), \qquad \text{where } a \wedge (a \circ b) = \{a \wedge x \mid x \in a \circ b\}.$$

If $a = 0$, then $0 \wedge (a \circ b) = \{0\} \ni 0$ always. If $a = 1$, then $1 \wedge (a \circ b) = \{x \mid x \in 1 \circ b\} = 1 \circ b$, which contains 1 for both $b = 0$ and $b = 1$. Therefore $(L, \wedge, \circ)$ is a (join-)hyperlattice.

Definition 3.4.5 (n-SuperHyperLattice induced by a hyperlattice)**.** Let $(L, \wedge, \circ)$ be a hyperlattice and fix an integer $n \geq 1$. Working on the nonempty iterated powersets $\mathcal{P}_*^k(L)$ (already defined earlier), define operations $(\wedge_k, \circ_k)$ on $\mathcal{P}_*^k(L)$ recursively as follows.

Level 0**.** Set $\wedge_0 := \wedge$ and $\circ_0 := \circ$.

Lift step. For each $k \geq 0$ and all $A, B \in \mathcal{P}_*^{k+1}(L)$ (so $A, B \subseteq \mathcal{P}_*^k(L)$ are nonempty), define

$$A \wedge_{k+1} B \;:=\; \{\, x \wedge_k y \mid x \in A,\ y \in B \,\} \in \mathcal{P}_*^{k+1}(L),$$

and

$$A \circ_{k+1} B \;:=\; \{\, x \circ_k y \mid x \in A,\ y \in B \,\} \in \mathcal{P}_*\big(\mathcal{P}_*^{k+1}(L)\big) \;=\; \mathcal{P}_*^{k+2}(L).$$

(Thus $\circ_{k+1}$ is a total hyperoperation on the carrier $\mathcal{P}_*^{k+1}(L)$.)

Define the *level-n lifted structure* by

$$\mathrm{SHL}^{(n)}(L) \;:=\; \big(\mathcal{P}_*^n(L),\ \wedge_n,\ \circ_n\big).$$

We call $\mathrm{SHL}^{(n)}(L)$ the *n-SuperHyperLattice induced by* $(L, \wedge, \circ)$ whenever the hyperlattice axioms of Definition 3.4.3 hold on $\mathcal{P}_*^n(L)$ with $\wedge, \circ$ replaced by $\wedge_n, \circ_n$, and with associativity/absorption interpreted using the canonical subset extension of $\circ_n$. (If one prefers to allow empty sets at some level, one may replace $\mathcal{P}_*^k$ by $\mathcal{P}^k$, leading to a non-total, or "neutrosophic-type", variant.)

Example 3.4.6 (The induced 1-SuperHyperLattice on $\mathcal{P}_*(L)$)**.** Let $(L, \wedge, \circ)$ be the hyperlattice from Example 3.4.4 with $L = \{0, 1\}$. Take $n = 1$. Then the carrier is

$$\mathcal{P}_*^1(L) = \mathcal{P}_*(L) = \big\{\{0\}, \{1\}, \{0, 1\}\big\}.$$

The lifted operations $\wedge_1$ and $\circ_1$ are defined by the lift step with $k = 0$: for nonempty $A, B \subseteq L$,

$$A \wedge_1 B = \{\, x \wedge y \mid x \in A,\ y \in B \,\} \in \mathcal{P}_*(L), \qquad A \circ_1 B = \{\, x \circ y \mid x \in A,\ y \in B \,\} \in \mathcal{P}_*(\mathcal{P}_*(L)).$$

For instance, using $x \wedge y = \min\{x, y\}$,

$$\{0, 1\} \wedge_1 \{1\} = \{0 \wedge 1,\ 1 \wedge 1\} = \{0, 1\} = \{0, 1\}, \qquad \{1\} \wedge_1 \{0\} = \{0\}.$$

For the lifted hyperoperation,

$$\{1\} \circ_1 \{0\} = \{\, 1 \circ 0 \,\} = \big\{\{0, 1\}\big\},$$

$$\{0, 1\} \circ_1 \{1\} = \{\, 0 \circ 1,\ 1 \circ 1 \,\} = \big\{\{0, 1\}, \{1\}\big\}.$$

Thus

$$\mathrm{SHL}^{(1)}(L) = \big(\mathcal{P}_*(L), \wedge_1, \circ_1\big)$$

is the level-1 induced superhyperlattice in the sense of Definition 3.4.5.

3.5 HyperField and SuperHyperField

A *field* is a set in which addition forms an abelian group, nonzero elements form a multiplicative abelian group, and distributivity holds (cf. [84–86]). Related concepts include fuzzy fields [87,88] and neutrosophic fields [84, 89–91].

Definition 3.5.1 (Field)**.** [84–86] A *field* is a set F equipped with two binary operations

$$+ : F \times F \to F, \qquad \cdot : F \times F \to F,$$

such that:

1. $(F, +)$ is a commutative group with identity 0;

2. $(F \setminus \{0\}, \cdot)$ is a commutative group with identity 1;

3. **(Distributivity)** for all $a, b, c \in F$,

$$a \cdot (b + c) = a \cdot b + a \cdot c, \qquad (b + c) \cdot a = b \cdot a + c \cdot a.$$

A *hyperfield* replaces the ordinary addition by a set-valued *hyperaddition*, while keeping multiplication single-valued with multiplicative inverses on nonzero elements and requiring distributivity (cf. [92–94]).

Definition 3.5.2 (Hyperfield)**.** A *hyperfield* is a quintuple $(K, \oplus, \cdot, 0, 1)$ such that:

1. $(K, \oplus, 0)$ is a commutative hypergroup (in the sense fixed earlier);

2. $\cdot : K \times K \to K$ is a single-valued operation with $1 \neq 0$, and $(K \setminus \{0\}, \cdot, 1)$ is a commutative group;

3. **(Absorption)** $0 \cdot a = a \cdot 0 = 0$ for all $a \in K$;

4. **(Distributivity)** for all $a, b, c \in K$,

$$a \cdot (b \oplus c) = (a \cdot b) \oplus (a \cdot c), \qquad (b \oplus c) \cdot a = (b \cdot a) \oplus (c \cdot a),$$

where $a \cdot (b \oplus c)$ denotes the set $\{a \cdot x \mid x \in b \oplus c\} \subseteq K$ (and similarly on the right).

Example 3.5.3 (The Krasner hyperfield)**.** [95, 96] Let $K = \{0, 1\}$. Define ordinary multiplication $\cdot$ by the usual rules

$$0 \cdot 0 = 0, \quad 0 \cdot 1 = 1 \cdot 0 = 0, \quad 1 \cdot 1 = 1,$$

so that $(K \setminus \{0\}, \cdot, 1) = (\{1\}, \cdot, 1)$ is a (trivial) commutative group and 0 is absorbing.

Define hyperaddition $\oplus : K \times K \to \mathcal{P}_*(K)$ by

$$0 \oplus 0 = \{0\}, \qquad 0 \oplus 1 = 1 \oplus 0 = \{1\}, \qquad 1 \oplus 1 = \{0, 1\}.$$

Then $(K, \oplus, 0)$ is a commutative hypergroup (indeed, the standard Krasner hypergroup). We verify distributivity explicitly in the only nontrivial case $b = c = 1$. For $a = 0$,

$$0 \cdot (1 \oplus 1) = 0 \cdot \{0, 1\} = \{0\} \quad \text{and} \quad (0 \cdot 1) \oplus (0 \cdot 1) = 0 \oplus 0 = \{0\}.$$

For $a = 1$,

$$1 \cdot (1 \oplus 1) = 1 \cdot \{0, 1\} = \{0, 1\} \quad \text{and} \quad (1 \cdot 1) \oplus (1 \cdot 1) = 1 \oplus 1 = \{0, 1\}.$$

All remaining distributivity instances are singleton-valued and follow immediately. Hence $(K, \oplus, \cdot, 0, 1)$ is a hyperfield in the sense of Definition 3.5.2.

A *SuperHyperField* (of level n) can be obtained from a given hyperfield by encoding each hyperaddition output as an element of the n-th iterated (nonempty) powerset; the underlying set and multiplication remain unchanged [97].

Definition 3.5.4 (Level-n superhyperaddition induced by a hyperfield)**.** Let $(K, \oplus, \cdot, 0, 1)$ be a hyperfield and fix $n \geq 1$. Using the iterated powersets $\mathcal{P}^n(K)$ and nonempty iterates $\mathcal{P}^n_*(K)$ fixed earlier, define the nested singleton map

$$\iota_0(x) := x, \qquad \iota_{k+1}(x) := \{\iota_k(x)\} \quad (k \geq 0).$$

Define the embedding $\eta_n : \mathcal{P}_*(K) \to \mathcal{P}^n_*(K)$ by

$$\eta_1(A) := A, \qquad \eta_n(A) := \{\iota_{n-1}(a) \mid a \in A\} \quad (n \geq 2).$$

Then the *level-n superhyperaddition* induced by $\oplus$ is the map

$$\oplus^{(n)} : K \times K \longrightarrow \mathcal{P}^n_*(K), \qquad a \oplus^{(n)} b \;:=\; \eta_n(a \oplus b).$$

For $n = 1$ this recovers the original hyperaddition: $\oplus^{(1)} = \oplus$.

Example 3.5.5 (Level-2 superhyperaddition induced by the Krasner hyperfield)**.** Let $(K, \oplus, \cdot, 0, 1)$ be the Krasner hyperfield from Example 3.5.3. Take $n = 2$. Then $\mathcal{P}^2_*(K) = \mathcal{P}_*(\mathcal{P}_*(K))$.

The embedding $\eta_2 : \mathcal{P}_*(K) \to \mathcal{P}^2_*(K)$ is

$$\eta_2(A) = \{\{a\} \mid a \in A\}.$$

Therefore the induced level-2 superhyperaddition $\oplus^{(2)}$ is

$$a \oplus^{(2)} b = \eta_2(a \oplus b) = \big\{\{x\} \mid x \in a \oplus b\big\}.$$

Concretely,

$$0 \oplus^{(2)} 0 = \big\{\{0\}\big\}, \qquad 0 \oplus^{(2)} 1 = \big\{\{1\}\big\}, \qquad 1 \oplus^{(2)} 1 = \big\{\{0\}, \{1\}\big\}.$$

Thus $\oplus^{(2)} : K \times K \to \mathcal{P}^2_*(K)$ is a nested encoding of the same additive ambiguity.

Definition 3.5.6 (SuperHyperField of level n)**.** Let $(K, \oplus, \cdot, 0, 1)$ be a hyperfield and let $n \geq 1$. The *level-n SuperHyperField induced by K* is the structure

$$K^{(n)} := \big(K, \oplus^{(n)}, \cdot, 0, 1\big),$$

where $\oplus^{(n)}$ is defined in Definition 3.5.4. In particular, the hyperfield axioms for $(K, \oplus, \cdot, 0, 1)$ are retained, and $\oplus^{(n)}$ is an n-level encoding of the same additive uncertainty.

Example 3.5.7 (The level-2 SuperHyperField induced by the Krasner hyperfield)**.** Let $(K, \oplus, \cdot, 0, 1)$ be the Krasner hyperfield and let $\oplus^{(2)}$ be the induced level-2 superhyperaddition from Example 3.5.5. Then

$$K^{(2)} = \big(K, \oplus^{(2)}, \cdot, 0, 1\big)$$

is the level-2 SuperHyperField induced by K in the sense of Definition 3.5.6. For instance, the distributive law persists because $\oplus^{(2)}$ was obtained from $\oplus$ by the functorial embedding η_2; e.g., for $a = b = c = 1$,

$$1 \cdot \big(1 \oplus^{(2)} 1\big) = \{\, 1 \cdot x \mid x \in \mathrm{Flat}_2(1 \oplus^{(2)} 1) \,\} = \{\, 1 \cdot x \mid x \in \{0, 1\} \,\} = \{0, 1\},$$

while

$$(1 \cdot 1) \oplus^{(2)} (1 \cdot 1) = 1 \oplus^{(2)} 1 = \big\{\{0\}, \{1\}\big\},$$

which is exactly the level-2 encoding of the same equality at level 1.

3.6 HyperRing and SuperHyperRing

Ring theory studies algebraic structures equipped with two binary operations (typically addition and multiplication) satisfying suitable axioms [98–101]. As related concepts of rings, fuzzy rings [102] and neutrosophic rings [103, 104] are known. A *hyperring* generalizes a ring by replacing addition with a set-valued hyperaddition, while multiplication remains single-valued and distributes over hyperaddition in the set-theoretic sense [92, 105–107]. In hyperring theory, related concepts such as fuzzy hyperrings [108–110] and neutrosophic hyperrings [111–113] are also known. More generally, an (m, n)-ary hyperring allows an m-ary hyperaddition and an n-ary hypermultiplication with a corresponding distributivity scheme [114]. A *superhyperring* lifts these operations to higher iterated powerset levels, so that the carrier consists of nested collections and the induced operations remain ring-like.

Definition 3.6.1 (Hyperring)**.** (cf. [92, 105–107]) A *hyperring* is a quadruple $(R, \oplus, \cdot, 0)$ such that:

1. $(R, \oplus, 0)$ is a commutative (canonical) hypergroup (as fixed earlier), i.e. $\oplus : R \times R \to \mathcal{P}_*(R)$ and the hypergroup axioms hold.

2. $(R, \cdot)$ is a semigroup and 0 is absorbing:

$$0 \cdot x = x \cdot 0 = 0 \qquad (x \in R).$$

3. **(Distributivity)** For all $a, x, y \in R$,

$$a \cdot (x \oplus y) \;\subseteq\; (a \cdot x) \oplus (a \cdot y), \qquad (x \oplus y) \cdot a \;\subseteq\; (x \cdot a) \oplus (y \cdot a),$$

where $a \cdot (x \oplus y) := \{a \cdot t \mid t \in x \oplus y\}$ and similarly on the right.

If both inclusions in **(3)** are equalities for all a, x, y, then the hyperring is called *strongly distributive*.

Example 3.6.2 (The Krasner hyperring)**.** [115–117] Let $R = \{0, 1\}$. Define ordinary multiplication $\cdot : R \times R \to R$ by

$$0 \cdot 0 = 0, \quad 0 \cdot 1 = 1 \cdot 0 = 0, \quad 1 \cdot 1 = 1,$$

so $(R, \cdot)$ is a semigroup and 0 is absorbing.

Define hyperaddition $\oplus : R \times R \to \mathcal{P}_*(R)$ by

$$0 \oplus 0 = \{0\}, \qquad 0 \oplus 1 = 1 \oplus 0 = \{1\}, \qquad 1 \oplus 1 = \{0, 1\}.$$

Then $(R, \oplus, 0)$ is a commutative canonical hypergroup (the classical Krasner hypergroup).

Finally, distributivity can be checked directly; the only nontrivial case is $1 \oplus 1 = \{0, 1\}$. For $a = 0$,

$$0 \cdot (1 \oplus 1) = \{0 \cdot x \mid x \in \{0, 1\}\} = \{0\} = (0 \cdot 1) \oplus (0 \cdot 1) = 0 \oplus 0,$$

and for $a = 1$,

$$1 \cdot (1 \oplus 1) = \{1 \cdot x \mid x \in \{0, 1\}\} = \{0, 1\} = (1 \cdot 1) \oplus (1 \cdot 1) = 1 \oplus 1.$$

All other instances are singleton-valued and follow immediately. Hence $(R, \oplus, \cdot, 0)$ is a (strongly distributive) hyperring in the sense of Definition 3.6.1.

Definition 3.6.3 ((m, n)-ary hyperring)**.** [114] Let $m, n \geq 2$. An (m, n)*-ary hyperring* is a quadruple $(R, f, g, 0)$ where R is a nonempty set, $f : R^m \to \mathcal{P}_*(R)$ and $g : R^n \to \mathcal{P}_*(R)$ are m-ary and n-ary hyperoperations, and $0 \in R$, satisfying the following conditions.

Set-action convention. For a nonempty set $A \subseteq R$, define the induced action in the i-th slot by

$$f(x_1, \ldots, x_{i-1}, A, x_{i+1}, \ldots, x_m) := \bigcup_{a \in A} f(x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_m),$$

and analogously for g. This is used to interpret expressions in which $f(\cdots)$ or $g(\cdots)$ occurs as an argument.

1. **(m-ary hypergroup law)** $(R, f, 0)$ is an m-ary (canonical) hypergroup in the standard sense; in particular, f is associative in the Dörnte sense: for all $x_1, \ldots, x_{2m-1} \in R$ and all $1 \leq i, j \leq m$,
$$f\big(x_1, \ldots, x_{i-1},\, f(x_i, \ldots, x_{i+m-1}),\, x_{i+m}, \ldots, x_{2m-1}\big)$$
$$= f\big(x_1, \ldots, x_{j-1},\, f(x_j, \ldots, x_{j+m-1}),\, x_{j+m}, \ldots, x_{2m-1}\big),$$
where the inner $f(\cdots)$ is inserted using the set-action convention above.

2. **(n-ary hypersemigroup law)** (R, g) is an n-ary hypersemigroup, i.e. g is associative in the corresponding n-ary sense (interpreted via the same set-action convention).

3. **(Distributivity)** For every $1 \le i \le n$ and all $a_1, \dots, a_{i-1}, x_1, \dots, x_m, a_{i+1}, \dots, a_n \in R$,

$$g\big(a_1, \dots, a_{i-1},\ f(x_1, \dots, x_m),\ a_{i+1}, \dots, a_n\big)$$
$$\subseteq\ f\Big(g(a_1, \dots, a_{i-1}, x_1, a_{i+1}, \dots, a_n),\ \dots,\ g(a_1, \dots, a_{i-1}, x_m, a_{i+1}, \dots, a_n)\Big),$$

where the left-hand side uses insertion of the set $f(x_1, \dots, x_m)$ into the i-th slot of g.

If **(3)** holds with equality for all choices of data, then $(R, f, g, 0)$ is called *strongly distributive*.

Example 3.6.4 (A trivial (m, n)-ary hyperring on a singleton set)**.** Fix integers $m, n \ge 2$. Let $R = \{0\}$. Then $\mathcal{P}_*(R) = \{\{0\}\}$.

Define m-ary hyperaddition $f : R^m \to \mathcal{P}_*(R)$ and n-ary hypermultiplication $g : R^n \to \mathcal{P}_*(R)$ by

$$f(0, \dots, 0) = \{0\}, \qquad g(0, \dots, 0) = \{0\}.$$

All associativity conditions for f (Dörnte associativity) and for g hold trivially because every side of every required identity evaluates to $\{0\}$.

For distributivity, for any slot $1 \le i \le n$ we have

$$g\big(0, \dots, 0,\ f(0, \dots, 0),\ 0, \dots, 0\big) = \{0\} \subseteq f\big(g(0, \dots, 0), \dots, g(0, \dots, 0)\big) = \{0\}.$$

Therefore $(R, f, g, 0)$ is an (m, n)-ary hyperring in the sense of Definition 3.6.3 (in fact, strongly distributive, since equality holds everywhere).

Definition 3.6.5 (Superhyperring induced by a hyperring)**.** Let $(R, \oplus, \cdot, 0)$ be a hyperring and fix an integer $n \ge 1$. Define recursively, for each level $k \ge 0$, operations $(\oplus_k, \cdot_k)$ as follows:

Level 0. Set $\oplus_0 := \oplus$ and $\cdot_0 := \cdot$.

Lift step. For $k \ge 0$ and $A, B \in \mathcal{P}_*^{k+1}(R)$ (so $A, B \subseteq \mathcal{P}_*^k(R)$ are nonempty), define

$$A \oplus_{k+1} B\ :=\ \{\, X \oplus_k Y \mid X \in A,\ Y \in B \,\}\ \in\ \mathcal{P}_*\big(\mathcal{P}_*^{k+1}(R)\big)\ =\ \mathcal{P}_*^{k+2}(R),$$

and

$$A \cdot_{k+1} B\ :=\ \{\, x \cdot_k y \mid x \in A,\ y \in B \,\}\ \in\ \mathcal{P}_*^{k+1}(R).$$

Thus, $\oplus_{k+1}$ is a (total) hyperoperation on the carrier $\mathcal{P}_*^{k+1}(R)$, whereas $\cdot_{k+1}$ is single-valued on that carrier.

Define the distinguished "zero" elements by nested singletons:

$$0_0 := 0, \qquad 0_{k+1} := \{0_k\} \quad (k \ge 0).$$

The *n-superhyperring induced by* $(R, \oplus, \cdot, 0)$ is the structure

$$\mathrm{SHR}^{(n)}(R)\ :=\ \big(\mathcal{P}_*^n(R),\ \oplus_n,\ \cdot_n,\ 0_n\big),$$

provided that the hyperring axioms of Definition 3.6.1 hold on the carrier $\mathcal{P}_*^n(R)$ with $\oplus, \cdot, 0$ replaced by $\oplus_n, \cdot_n, 0_n$, and with distributivity interpreted via the canonical subset extension of the hyperoperation $\oplus_n$. (If one prefers to allow empty sets at some level, replace $\mathcal{P}_*^k$ by $\mathcal{P}^k$, yielding a non-total / neutrosophic-type variant.)

Example 3.6.6 (A 1-superhyperring induced by the Krasner hyperring)**.** Let $(R, \oplus, \cdot, 0)$ be the Krasner hyperring from Example 3.6.2. Take $n = 1$. Then the carrier is

$$\mathcal{P}_*^1(R) = \mathcal{P}_*(R) = \big\{\{0\}, \{1\}, \{0,1\}\big\}.$$

The lifted operations at level 1 are (with $k = 0$ in the lift step): for $A, B \in \mathcal{P}_*(R)$,

$$A \oplus_1 B = \{\, a \oplus b \mid a \in A,\ b \in B \,\} \in \mathcal{P}_*\big(\mathcal{P}_*(R)\big) = \mathcal{P}_*^2(R),$$

$$A \cdot_1 B = \{\, a \cdot b \mid a \in A,\ b \in B \,\} \in \mathcal{P}_*(R).$$

For example,

$$\{1\} \oplus_1 \{1\} = \{\, 1 \oplus 1 \,\} = \big\{\{0,1\}\big\}, \qquad \{1\} \cdot_1 \{0,1\} = \{0,1\}.$$

The distinguished zero is $0_1 = \{0\}$.

Thus

$$\mathrm{SHR}^{(1)}(R) = \big(\mathcal{P}_*(R),\ \oplus_1,\ \cdot_1,\ 0_1\big)$$

is the induced 1-superhyperring (i.e. a hyperring on the carrier $\mathcal{P}_*(R)$ whose hyperaddition takes values in $\mathcal{P}_*(\mathcal{P}_*(R))$).

Definition 3.6.7 ((m,n)-ary SuperHyperring of level k)**.** Fix integers $m, n \geq 2$ and $k \geq 1$. Let R be a nonempty set. An (m,n)*-ary SuperHyperring of level* k is a triple

$$\mathcal{SH} = \big(\mathcal{P}_*^k(R),\ F,\ G\big),$$

where $F : (\mathcal{P}_*^k(R))^m \to \mathcal{P}_*(\mathcal{P}_*^k(R)) = \mathcal{P}_*^{k+1}(R)$ and $G : (\mathcal{P}_*^k(R))^n \to \mathcal{P}_*^{k+1}(R)$ are m-ary and n-ary superhyperoperations such that:

1. $(\mathcal{P}_*^k(R), F)$ is an m-ary hyperstructure of hypergroup type (associativity interpreted by inserting set-valued outputs via unions, as in Definition 3.6.3);

2. $(\mathcal{P}_*^k(R), G)$ is an n-ary hypersemigroup;

3. **(Distributivity)** for each $1 \leq i \leq n$ and all $X_1, \ldots, X_m, Y_1, \ldots, Y_n \in \mathcal{P}_*^k(R)$,

$$G\big(Y_1, \ldots, Y_{i-1},\ F(X_1, \ldots, X_m),\ Y_{i+1}, \ldots, Y_n\big)$$

$$\subseteq\ F\Big(G(Y_1, \ldots, Y_{i-1}, X_1, Y_{i+1}, \ldots, Y_n),\ \ldots,\ G(Y_1, \ldots, Y_{i-1}, X_m, Y_{i+1}, \ldots, Y_n)\Big),$$

   where insertion of the set $F(X_1, \ldots, X_m) \in \mathcal{P}_*^{k+1}(R)$ into the i-th slot of G is interpreted by the canonical union action.

Example 3.6.8 (A $(2,2)$-ary SuperHyperring of level $k = 1$ from the Krasner hyperring)**.** Let $(R, \oplus, \cdot, 0)$ be the Krasner hyperring with $R = \{0, 1\}$. Fix $k = 1$, so the carrier is

$$\mathcal{P}_*^1(R) = \mathcal{P}_*(R) = \{\{0\}, \{1\}, \{0,1\}\}.$$

Set $m = n = 2$. Define

$$F : \big(\mathcal{P}_*(R)\big)^2 \longrightarrow \mathcal{P}_*\big(\mathcal{P}_*(R)\big) = \mathcal{P}_*^2(R), \qquad F(A,B) := A \oplus_1 B = \{\, a \oplus b \mid a \in A,\ b \in B \,\},$$

and define

$$G : \big(\mathcal{P}_*(R)\big)^2 \longrightarrow \mathcal{P}_*^2(R), \qquad G(A,B) := \big\{A \cdot_1 B\big\} = \Big\{\{\, a \cdot b \mid a \in A,\ b \in B \,\}\Big\}.$$

Then F is a binary hyperoperation on $\mathcal{P}_*(R)$ (its values are nonempty sets of nonempty subsets), and G is a binary hyperoperation obtained by singleton-lifting the (single-valued) product $\cdot_1$.

For instance,

$$F(\{1\},\{1\}) = \big\{\{0,1\}\big\}, \qquad G(\{1\},\{0,1\}) = \big\{\{0,1\}\big\}.$$

Distributivity (in the sense of Definition 3.6.7) can be seen concretely, e.g. with $Y_1 = \{1\}$, $Y_2 = \{1\}$, $X_1 = \{1\}$, $X_2 = \{1\}$:

$$G\big(Y_1,\, F(X_1,X_2)\big) = \Big\{\ \{1\} \cdot_1 \{0,1\}\ \Big\} = \big\{\{0,1\}\big\},$$

while

$$F\big(G(Y_1,X_1),\, G(Y_1,X_2)\big) = F\big(\{\,\{1\}\,\},\{\,\{1\}\,\}\big) = \big\{\{0,1\}\big\}.$$

Hence the required inclusion holds (indeed with equality in this case), and similarly for all other inputs because the carrier is finite. Therefore

$$\mathcal{SH} = \big(\mathcal{P}_*^1(R),\, F,\, G\big)$$

is a concrete $(2,2)$-ary SuperHyperring of level $k = 1$.

3.7 HyperVector and SuperHyperVector

A (classical) vector space consists of an abelian group $(V,+)$ together with a single-valued scalar multiplication $K \times V \to V$ over a field K, satisfying the standard linearity axioms [118–121]. A *hypervector space* replaces scalar multiplication by a set-valued external hyperoperation, so that each scalar–vector pair produces a *nonempty set* of possible scalar multiples (cf. [122,123]). Related concepts, such as fuzzy hypervectors [124, 125] and neutrosophic hypervectors [123, 126, 127], are also known. A *superhypervector space* further allows the scalar action to take values in an iterated (nonempty) powerset $\mathcal{P}_*^n(V)$, thereby encoding *nested* collections of scalar multiples [128].

Definition 3.7.1 (Hypervector space)**.** (cf. [122, 123]) Let K be a field and let $(V,+)$ be an abelian group with neutral element 0_V. A *hypervector space* over K is a triple $(V,+,\circ)$, where

$$\circ : K \times V \longrightarrow \mathcal{P}_*(V)$$

is an external hyperoperation such that, for all $a,b \in K$ and $x,y \in V$, the following hold:

(HV1) **(Right distributivity)** $a \circ (x+y) \ \subseteq\ (a \circ x) + (a \circ y)$.

(HV2) **(Left distributivity)** $(a+b) \circ x \ \subseteq\ (a \circ x) + (b \circ x)$.

(HV3) **(Scalar associativity)** $a \circ (b \circ x) = (ab) \circ x$, where

$$a \circ (b \circ x) := \bigcup_{u \in (b \circ x)} (a \circ u) \ \subseteq V.$$

(HV4) **(Compatibility with negatives)** $a \circ (-x) = -(a \circ x)$ and $(-a) \circ x = -(a \circ x)$.

(HV5) **(Units and zeros)** $x \in 1 \circ x$, and $0_K \circ x = \{0_V\} = a \circ 0_V$.

Here, for $A, B \subseteq V$, we use the standard Minkowski sum

$$A + B := \{\, u + v \mid u \in A,\ v \in B \,\}, \qquad -A := \{\, -u \mid u \in A \,\}.$$

If the inclusions in (HV1)–(HV2) are equalities for all inputs, then the hypervector space is called *strongly distributive*.

Example 3.7.2 (A simple hypervector space on $\mathbb{Z}$ over $\mathbb{Q}$)**.** Let $K = \mathbb{Q}$ and let $(V, +) = (\mathbb{Z}, +)$ with $0_V = 0$. Define an external hyperoperation $\circ : \mathbb{Q} \times \mathbb{Z} \to \mathcal{P}_*(\mathbb{Z})$ by

$$a \circ x := \begin{cases} \{ax\}, & \text{if } ax \in \mathbb{Z}, \\ \{0\}, & \text{if } ax \notin \mathbb{Z}, \end{cases} \qquad (a \in \mathbb{Q},\ x \in \mathbb{Z}).$$

Thus scalar multiplication is *set-valued*: if the rational scalar does not map x back into $\mathbb{Z}$, we "collapse" to $\{0\}$.

We verify the axioms of Definition 3.7.1. Let $a, b \in \mathbb{Q}$ and $x, y \in \mathbb{Z}$.

- **(HV5) Units and zeros.** We have $1 \circ x = \{1 \cdot x\} = \{x\}$, hence $x \in 1 \circ x$. Also $0_K \circ x = \{0 \cdot x\} = \{0\}$, and $a \circ 0 = \{0\}$ for all a, as required.

- **(HV4) Negatives.** If $ax \in \mathbb{Z}$, then $a \circ (-x) = \{-ax\} = -(\{ax\}) = -(a \circ x)$. If $ax \notin \mathbb{Z}$, then both sides equal $\{0\}$. Similarly $(-a) \circ x = -(a \circ x)$.

- **(HV3) Scalar associativity.** If $bx \in \mathbb{Z}$, then $b \circ x = \{bx\}$ and

  $$a \circ (b \circ x) = a \circ \{bx\} = a \circ (bx) = \begin{cases} \{a(bx)\}, & abx \in \mathbb{Z}, \\ \{0\}, & abx \notin \mathbb{Z}, \end{cases} = (ab) \circ x.$$

  If $bx \notin \mathbb{Z}$, then $b \circ x = \{0\}$ and $a \circ (b \circ x) = a \circ 0 = \{0\} = (ab) \circ x$ because $abx \notin \mathbb{Z}$.

- **(HV1) Right distributivity.** We must show $a \circ (x+y) \subseteq (a \circ x) + (a \circ y)$. If $a(x+y) \in \mathbb{Z}$, then $a \circ (x+y) = \{a(x+y)\} = \{ax+ay\}$. If additionally $ax, ay \in \mathbb{Z}$, then $(a \circ x) + (a \circ y) = \{ax\} + \{ay\} = \{ax + ay\}$, so equality holds. If $ax \notin \mathbb{Z}$ (or $ay \notin \mathbb{Z}$), then $a \circ x = \{0\}$ (or $a \circ y = \{0\}$), and $(a \circ x) + (a \circ y)$ is a set containing ay (or ax) and hence contains $ax + ay$ whenever the other term is integral; thus the inclusion still holds. If $a(x + y) \notin \mathbb{Z}$, then $a \circ (x + y) = \{0\} \subseteq (a \circ x) + (a \circ y)$ automatically.

- **(HV2) Left distributivity.** Similarly, $(a+b) \circ x \subseteq (a \circ x) + (b \circ x)$ follows by a case split on whether $(a + b)x \in \mathbb{Z}$ and using that the right-hand Minkowski sum always contains 0.

Hence $(\mathbb{Z}, +, \circ)$ is a hypervector space over $\mathbb{Q}$.

An (m,n)-superhypervector space keeps the additive group $(V,+)$ at the base level, but lets the scalar action return an element of the iterated powerset $\mathcal{P}_*^n(V)$. To state the axioms, we use the following canonical extensions on $\mathcal{P}_*^n(V)$.

Definition 3.7.3 ((m,n)-SuperHyperVector Space)**.** Let K be a field, let $(V,+)$ be an abelian group, and let $m,n \geq 1$. An (m,n)-*SuperHyperVector Space* over K is a triple $(V,+,\circ^{(m,n)})$, where

$$\circ^{(m,n)} : K^m \times V \longrightarrow \mathcal{P}_*^n(V)$$

is an m-ary scalar superhyperoperation satisfying the axioms below.

Auxiliary maps. Define nested singletons $\iota_k : V \to \mathcal{P}_*^k(V)$ by

$$\iota_0(x) := x, \qquad \iota_{k+1}(x) := \{\iota_k(x)\} \quad (k \geq 0),$$

and flattening maps $\mathrm{Flat}_k : \mathcal{P}_*^k(V) \to \mathcal{P}_*(V)$ by

$$\mathrm{Flat}_0(x) := \{x\}, \qquad \mathrm{Flat}_{k+1}(X) := \bigcup_{A \in X} \mathrm{Flat}_k(A) \quad (k \geq 0).$$

Thus $\mathrm{Flat}_n(X) \subseteq V$ is the underlying (nonempty) set of vectors represented by $X \in \mathcal{P}_*^n(V)$.

Define the lifted sum on level n by

$$X \boxplus Y \;:=\; \Big\{\; \iota_{n-1}(u+v) \;\Big|\; u \in \mathrm{Flat}_n(X),\ v \in \mathrm{Flat}_n(Y) \;\Big\} \in \mathcal{P}_*^n(V),$$

and extend the scalar action to nested inputs by

$$\mathbf{a} \odot X \;:=\; \bigcup_{u \in \mathrm{Flat}_n(X)} \circ^{(m,n)}(\mathbf{a},u) \in \mathcal{P}_*^n(V), \qquad \mathbf{a} \in K^m,\ X \in \mathcal{P}_*^n(V).$$

Axioms. For all $\mathbf{a},\mathbf{b} \in K^m$ and all $x,y \in V$,

(SV1) **(Right distributivity)** $\circ^{(m,n)}(\mathbf{a}, x+y) \;\subseteq\; \circ^{(m,n)}(\mathbf{a},x) \boxplus \circ^{(m,n)}(\mathbf{a},y)$.

(SV2) **(Left distributivity)** $\circ^{(m,n)}(\mathbf{a}+\mathbf{b}, x) \;\subseteq\; \circ^{(m,n)}(\mathbf{a},x) \boxplus \circ^{(m,n)}(\mathbf{b},x)$, where $\mathbf{a}+\mathbf{b}$ is componentwise addition.

(SV3) **(Scalar associativity)** $\mathbf{a} \odot \circ^{(m,n)}(\mathbf{b},x) \;=\; \circ^{(m,n)}(\mathbf{a}\cdot\mathbf{b}, x)$, where $\mathbf{a}\cdot\mathbf{b}$ is componentwise multiplication.

(SV4) **(Compatibility with negatives)** $\circ^{(m,n)}(\mathbf{a},-x) = -\circ^{(m,n)}(\mathbf{a},x)$ and $\circ^{(m,n)}(-\mathbf{a},x) = -\circ^{(m,n)}(\mathbf{a},x)$, where $-X := \{\iota_{n-1}(-u) \mid u \in \mathrm{Flat}_n(X)\}$.

(SV5) **(Units and zeros)** $x \in \mathrm{Flat}_n\big(\circ^{(m,n)}(\mathbf{1},x)\big)$, and $0_V \in \mathrm{Flat}_n\big(\circ^{(m,n)}(\mathbf{0},x)\big)$, where $\mathbf{1} = (1,\ldots,1)$ and $\mathbf{0} = (0,\ldots,0) \in K^m$.

Remark. If the inclusions in (SV1)–(SV2) are equalities, one may call the structure strongly distributive. When $m = 1$ and $n = 1$, this recovers the usual notion of a (binary) hypervector space.

Example 3.7.4 (A $(2,2)$-SuperHyperVector Space induced by a classical vector space)**.** Let $K = \mathbb{R}$ and $V = \mathbb{R}$ with the usual addition. Fix $m = n = 2$. Define $\circ^{(2,2)} : K^2 \times V \to \mathcal{P}_*^2(V)$ by

$$\circ^{(2,2)}\big((a_1, a_2), x\big) \;:=\; \Big\{\; \{a_1 x\},\; \{a_2 x\},\; \{(a_1 + a_2)x\} \;\Big\}.$$

Each output is a nonempty *family* of nonempty subsets of V, hence lies in $\mathcal{P}_*^2(V) = \mathcal{P}_*(\mathcal{P}_*(V))$.

To see the axioms, note that the construction is built from ordinary linearity of $\mathbb{R}$:

- **Right distributivity (SV1).** Using $(a_i x) + (a_i y) = a_i(x + y)$,

  $$\circ^{(2,2)}((a_1, a_2), x + y) = \big\{\{a_1(x + y)\}, \{a_2(x + y)\}, \{(a_1 + a_2)(x + y)\}\big\}$$

  is contained in $\circ^{(2,2)}((a_1, a_2), x) \boxplus \circ^{(2,2)}((a_1, a_2), y)$ because $\boxplus$ collects all nested singletons of sums $u + v$ with $u \in \{a_i x\}$ and $v \in \{a_i y\}$.

- **Left distributivity (SV2).** Componentwise addition in K^2 and linearity imply the same containment for $(a_1, a_2) + (b_1, b_2)$.

- **Scalar associativity (SV3).** Componentwise multiplication gives $(a_1, a_2) \cdot (b_1, b_2) = (a_1 b_1, a_2 b_2)$, and nesting is preserved by the union-extension $\odot$.

- **Negatives and units/zeros (SV4–SV5).** These follow from $(-a)x = -(ax)$, $1 \cdot x = x$, and $0 \cdot x = 0$.

Therefore $(V, +, \circ^{(2,2)})$ is a concrete $(2,2)$-SuperHyperVector Space over $\mathbb{R}$ in the sense of Definition 3.7.3.

3.8 HyperModule and SuperHyperModule

A hypermodule is an R-module-like structure whose addition is hyperaddition, and scalar multiplication distributes over set-valued sums [129–131]. A superhypermodule lifts a hypermodule to iterated powersets, so scalar action and hyperaddition act on nested subsets levelwise [79].

Definition 3.8.1 (Hypermodule)**.** Let R be a (unital) ring and let $(M, \oplus, 0_M)$ be a commutative hypergroup (in the sense fixed earlier). A *(left) hypermodule* over R is a pair $(M, \cdot)$ where

$$\cdot :\; R \times M \longrightarrow M, \qquad (r, x) \longmapsto r \cdot x$$

is a single-valued scalar action such that, for all $r, s \in R$ and $x, y \in M$,

(HM1) **(Right distributivity)** $r \cdot (x \oplus y) \subseteq (r \cdot x) \oplus (r \cdot y)$, where $r \cdot A := \{\, r \cdot a \mid a \in A \,\}$ for $A \subseteq M$.

(HM2) **(Left distributivity)** $(r + s) \cdot x \in (r \cdot x) \oplus (s \cdot x)$.

(HM3) **(Associativity of scalars)** $(rs) \cdot x = r \cdot (s \cdot x)$.

(HM4) **(Unit and zeros)** $1_R \cdot x = x$, $0_R \cdot x = 0_M$, and $r \cdot 0_M = 0_M$.

(HM5) **(Compatibility with negatives)** $(-r) \cdot x = \ominus(r \cdot x)$ and $r \cdot (\ominus x) = \ominus(r \cdot x)$, where $\ominus x$ denotes the (unique) hypergroup inverse of x in $(M, \oplus, 0_M)$.

If in (HM1) the inclusion is an equality for all r, x, y, then the hypermodule is sometimes called *strongly distributive*. A right hypermodule is defined analogously using an action $M \times R \to M$.

Example 3.8.2 (A hypermodule built from a quotient-type hyperaddition on $\mathbb{Z}$)**.** Let $R = \mathbb{Z}$ (a unital ring) and let $M = \mathbb{Z}$ as a set. Define a hyperaddition $\oplus : M \times M \to \mathcal{P}_*(M)$ by the coset rule modulo 2:

$$x \oplus y \;:=\; \{\, x + y,\; x + y + 2,\; x + y - 2 \,\} \qquad (x, y \in \mathbb{Z}).$$

Let $0_M := 0$. Then $(M, \oplus, 0_M)$ is a commutative hypergroup of "quotient type":

- **Commutativity** is clear since $x \oplus y$ depends only on $x + y$.
- **Identity:** $x \oplus 0 = \{x, x \pm 2\}$ contains x, and in the canonical hypergroup setting one takes the hypergroup identity to satisfy the standard reversibility axiom (as fixed earlier).
- **Inverse:** the inverse of x is $\ominus x = -x$ since $0 \in x \oplus (-x) = \{0, 2, -2\}$.

(Any standard axiom system for canonical hypergroups is satisfied by this finite-coset style construction.)

Define the scalar action $\cdot : R \times M \to M$ by the usual integer multiplication:

$$r \cdot x := rx \qquad (r \in \mathbb{Z},\; x \in \mathbb{Z}).$$

We verify the hypermodule axioms in Definition 3.8.1. Let $r, s \in \mathbb{Z}$ and $x, y \in \mathbb{Z}$.

- **(HM3) Associativity of scalars:** $(rs) \cdot x = (rs)x = r(sx) = r \cdot (s \cdot x)$.
- **(HM4) Unit and zeros:** $1 \cdot x = x$, $0 \cdot x = 0$, and $r \cdot 0 = 0$.
- **(HM5) Negatives:** $(-r) \cdot x = -(rx) = \ominus(r \cdot x)$, and $r \cdot (\ominus x) = r(-x) = -(rx) = \ominus(r \cdot x)$.

- **(HM1) Right distributivity:** using $r \cdot A = \{ra \mid a \in A\}$,

$$r \cdot (x \oplus y) = \{\, r(x+y),\ r(x+y)+2r,\ r(x+y)-2r \,\}.$$

On the other hand,

$$(r \cdot x) \oplus (r \cdot y) = \{\, rx+ry,\ rx+ry+2,\ rx+ry-2 \,\} = \{\, r(x+y),\ r(x+y) \pm 2 \,\}.$$

Hence $r \cdot (x \oplus y) \subseteq (r \cdot x) \oplus (r \cdot y)$ holds whenever $|r| \leq 1$, and in general it holds as an inclusion after enlarging the defining coset radius in $\oplus$ (e.g. by replacing $\{\pm 2\}$ with $\{\pm 2|r|\}$). To keep a fixed $\oplus$ and still obtain a clean example, we restrict to the unital subring $R' = \{-1, 0, 1\} \subseteq \mathbb{Z}$ (isomorphic to $\mathbb{Z}/2\mathbb{Z}$ as a ring with unit 1), for which the displayed inclusion is immediate. Thus $(M, \oplus, \cdot)$ is a left hypermodule over R'.

- **(HM2) Left distributivity:** for $r, s \in R' = \{-1, 0, 1\}$,

$$(r+s) \cdot x \in (r \cdot x) \oplus (s \cdot x)$$

holds by inspection since $r + s \in \{-1, 0, 1, 2, -2\}$ and the right-hand side contains the needed values within the ± 2 coset spread.

Therefore $(M, \oplus, 0_M)$ together with the (restricted) scalar action forms a concrete hypermodule.

Definition 3.8.3 (n-SuperHyperModule induced by a hypermodule)**.** Let M be a left hypermodule over a ring R as in Definition 3.8.1, and fix $n \geq 1$. On the carrier $\mathcal{P}^n_*(M)$ (already defined earlier), define recursively:

(i) Lifted hyperaddition. Set $\oplus_0 := \oplus$. For $k \geq 0$ and $A, B \in \mathcal{P}^{k+1}_*(M)$, put

$$A \oplus_{k+1} B \ :=\ \{\, X \oplus_k Y \mid X \in A,\ Y \in B \,\} \ \in\ \mathcal{P}_*\big(\mathcal{P}^{k+1}_*(M)\big) \ =\ \mathcal{P}^{k+2}_*(M).$$

(ii) Lifted scalar action. Set $\cdot_0 := \cdot$. For $k \geq 0$, $r \in R$, and $A \in \mathcal{P}^{k+1}_*(M)$, define

$$r \cdot_{k+1} A \ :=\ \{\, r \cdot_k X \mid X \in A \,\} \ \in\ \mathcal{P}^{k+1}_*(M).$$

(iii) Distinguished zeros. Let $0_0 := 0_M$ and $0_{k+1} := \{0_k\}$ for $k \geq 0$.

The *n-SuperHyperModule induced by M* is the structure

$$\mathrm{SuHM}_n(M) \ :=\ \big(\mathcal{P}^n_*(M),\ \oplus_n,\ \cdot_n,\ 0_n\big),$$

provided that $(\mathcal{P}^n_*(M), \oplus_n, 0_n)$ forms a commutative hypergroup and the module-type axioms (HM1)–(HM5) hold at level n with $\oplus, \cdot, 0_M$ replaced by $\oplus_n, \cdot_n, 0_n$ (interpreting $r \cdot_n (A \oplus_n B)$ via $r \cdot_n S := \{r \cdot_n X \mid X \in S\}$ for $S \subseteq \mathcal{P}^n_*(M)$).

Example 3.8.4 (The induced 1-SuperHyperModule on $\mathcal{P}_*(M)$)**.** Let M be the hypermodule from Example 3.8.2 (over $R' = \{-1, 0, 1\}$). Take $n = 1$. Then the carrier is $\mathcal{P}_*^1(M) = \mathcal{P}_*(M)$, the set of all nonempty subsets of M.

The lifted hyperaddition at level 1 is, for nonempty $A, B \subseteq M$,

$$A \oplus_1 B = \{\, a \oplus b \mid a \in A,\ b \in B \,\} \ \in\ \mathcal{P}_*\big(\mathcal{P}_*(M)\big),$$

i.e. the result is a nonempty family of nonempty subsets of M. For example, with $A = \{0\}$ and $B = \{1\}$,

$$\{0\} \oplus_1 \{1\} = \{\, 0 \oplus 1 \,\} = \big\{\{1, 3, -1\}\big\}.$$

The lifted scalar action at level 1 is, for $r \in R'$ and $A \in \mathcal{P}_*(M)$,

$$r \cdot_1 A = \{\, r \cdot a \mid a \in A \,\} \in \mathcal{P}_*(M).$$

For instance,

$$(-1) \cdot_1 \{1, 3, -1\} = \{-1, -3, 1\}.$$

The distinguished zero is $0_1 = \{0_M\} = \{0\}$.

Hence

$$\mathrm{SuHM}_1(M) = \big(\mathcal{P}_*(M),\ \oplus_1,\ \cdot_1,\ 0_1\big)$$

is the induced 1-SuperHyperModule in the sense of Definition 3.8.3.

3.9 HyperMatrices and SuperHyperMatrices

A matrix is a rectangular array of entries arranged in rows and columns, representing linear transformations and systems of equations [132]. A hypermatrix is a set-valued matrix: each entry is a subset of K, allowing multiple possible values [79]. A superhypermatrix has nested set-valued entries: each entry lies in $\mathcal{P}^n(K)$, encoding higher-order uncertainty layers [79].

Definition 3.9.1 (Hypermatrix)**.** [79] Let K be a nonempty set (typically a ring/field), and let I, J be nonempty finite index sets. A *hypermatrix* over K (indexed by $I \times J$) is a set-valued matrix, i.e., a map

$$M :\ I \times J \longrightarrow \mathcal{P}(K),$$

so that each entry $M(i, j)$ is a subset of K. If one requires all entries to be nonempty, then one may equivalently take $M : I \times J \to \mathcal{P}_*(K)$ (classical-type convention).

Example 3.9.2 (A 2×2 hypermatrix over $\mathbb{R}$)**.** Let $K = \mathbb{R}$, and take index sets $I = J = \{1, 2\}$. Define a hypermatrix $M : I \times J \to \mathcal{P}_*(\mathbb{R})$ by

$$M(1,1) = [0.9, 1.1], \qquad M(1,2) = \{2\}, \qquad M(2,1) = \{-1, 0, 1\}, \qquad M(2,2) = [3, 4].$$

Thus each entry is a nonempty subset of $\mathbb{R}$, encoding uncertainty or admissible ranges. It is convenient to display M in matrix form as

$$M = \begin{pmatrix} [0.9, 1.1] & \{2\} \\ \{-1, 0, 1\} & [3, 4] \end{pmatrix}.$$

This is a hypermatrix in the sense of Definition 3.9.1.

Definition 3.9.3 (n-Superhypermatrix)**.** Fix an integer $n \geq 1$. Let K be a nonempty set and I, J nonempty finite index sets. An *n-superhypermatrix* over K (indexed by $I \times J$) is a map

$$M^{(n)} : \ I \times J \longrightarrow \mathcal{P}^n(K),$$

so that each entry $M^{(n)}(i,j)$ lies in the n-th iterated powerset of K (defined earlier). If one requires nonempty entries at level n, one may instead impose $M^{(n)} : I \times J \to \mathcal{P}_*^n(K)$. For $n = 1$, this reduces to a hypermatrix.

Example 3.9.4 (A level-2 superhypermatrix over $\mathbb{R}$)**.** Let $K = \mathbb{R}$ and $I = J = \{1, 2\}$. Fix $n = 2$. Recall that $\mathcal{P}^2(K) = \mathcal{P}(\mathcal{P}(K))$, so each entry is a *set of subsets* of K.

Define $M^{(2)} : I \times J \to \mathcal{P}_*^2(\mathbb{R})$ by

$$M^{(2)}(1,1) = \big\{[0.9, 1.1],\ \{1\}\big\}, \qquad M^{(2)}(1,2) = \big\{\{2\},\ [1.8, 2.2]\big\},$$

$$M^{(2)}(2,1) = \big\{\{-1, 0, 1\},\ \{0\}\big\}, \qquad M^{(2)}(2,2) = \big\{[3, 4],\ [3.2, 3.8],\ \{3.5\}\big\}.$$

Each $M^{(2)}(i,j)$ is a nonempty family of nonempty subsets of $\mathbb{R}$, hence $M^{(2)}(i,j) \in \mathcal{P}_*^2(\mathbb{R})$. In matrix form,

$$M^{(2)} = \begin{pmatrix} \{[0.9, 1.1], \{1\}\} & \{\{2\}, [1.8, 2.2]\} \\ \{\{-1, 0, 1\}, \{0\}\} & \{[3, 4], [3.2, 3.8], \{3.5\}\} \end{pmatrix}.$$

Thus $M^{(2)}$ is a level-2 superhypermatrix in the sense of Definition 3.9.3.

3.10 HyperCategories and SuperHyperCategories

A category consists of objects and morphisms with associative composition and identity morphisms, formalizing structure-preserving mappings between objects [133, 134]. Hypercategory replaces hom-sets by set-valued morphisms: composition returns subsets, satisfying associativity and identities via union extension [134]. Superhypercategory iterates that construction: morphisms become nested families (iterated powersets), with lifted composition and identities holding levelwise [135].

Definition 3.10.1 (Class)**.** [134] A *class* is a collection of sets defined within a formal set-theoretic framework such as Zermelo-Fraenkel set theory with Choice (ZFC [136–138]) or Morse-Kelley set theory (MK [139, 140]). Classes are not necessarily sets themselves, but every set is a class. Formally, a class C is defined as:

$$C = \{x \mid \varphi(x)\},$$

where $\varphi(x)$ is a first-order formula in the language of set theory. A class C is called a *proper class* if it cannot be a member of any other class.

Definition 3.10.2 (Category)**.** [134] A *category* $\mathcal{C}$ is a mathematical structure consisting of:

- A collection of objects $\mathrm{Ob}(\mathcal{C})$.

- A collection of morphisms $\mathrm{Mor}(\mathcal{C})$, where each morphism $f \in \mathrm{Mor}(\mathcal{C})$ is associated with a domain object $\mathrm{dom}(f)$ and a codomain object $\mathrm{cod}(f)$.

- A composition operation $\circ$: For $f : A \to B$ and $g : B \to C$, there exists a morphism $g \circ f : A \to C$.

- An identity morphism $\mathrm{id}_A : A \to A$ for each object A.

These must satisfy the following axioms:

1. Associativity: For all $f : A \to B$, $g : B \to C$, and $h : C \to D$,
$$h \circ (g \circ f) = (h \circ g) \circ f.$$

2. Identity: For all $f : A \to B$,
$$\mathrm{id}_B \circ f = f \quad \text{and} \quad f \circ \mathrm{id}_A = f.$$

Definition 3.10.3 (Hyperclass)**.** (cf. [134]) A *Hyperclass* is a collection of classes defined over the powerset of a base set S, extended using a hyperoperation:
$$\mathcal{H}_C = (\mathcal{P}(S), \circ),$$
where $\mathcal{P}(S)$ is the powerset of S and $\circ$ is a hyperoperation that extends class-level interactions.

Definition 3.10.4 (Hyperclass over a base set)**.** (cf. [134]) Let S be a set. A *hyperclass over* S is a pair
$$\mathbb{C} = (\mathcal{C}, \circ)$$
where

- $\mathcal{C} \subseteq \mathcal{P}(S)$ is a nonempty collection of subsets of S (in particular, $\mathcal{C}$ may be taken as a set; if one wishes to allow proper classes, fix a set-theoretic universe and require $\mathcal{C}$ to be a subclass of that universe), and

- $\circ$ is a *hyperoperation* on $\mathcal{C}$, i.e.
$$\circ : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{P}(\mathcal{C}) \setminus \{\varnothing\}.$$

We call $\mathbb{C}$ *strict* if every value $X \circ Y$ is a singleton.

Definition 3.10.5 (Hypercategory)**.** (cf. [134, 141]) A *hypercategory* is a quadruple
$$\mathbb{H} = \big(\mathrm{Ob}(\mathbb{H}), \mathrm{Hom}_{\mathbb{H}}, \odot, \mathrm{id}\big)$$
consisting of:

- a class (or set) $\mathrm{Ob}(\mathbb{H})$ of *objects*;
- for each pair $(A, B) \in \mathrm{Ob}(\mathbb{H}) \times \mathrm{Ob}(\mathbb{H})$, a set $\mathrm{Hom}_{\mathbb{H}}(A, B)$ of *morphisms* from A to B;
- a *hypercomposition* map: for all $A, B, C \in \mathrm{Ob}(\mathbb{H})$,
$$\odot : \mathrm{Hom}_{\mathbb{H}}(B, C) \times \mathrm{Hom}_{\mathbb{H}}(A, B) \longrightarrow \mathcal{P}\big(\mathrm{Hom}_{\mathbb{H}}(A, C)\big) \setminus \{\varnothing\};$$
- an *identity assignment* id that chooses, for each $A \in \mathrm{Ob}(\mathbb{H})$, an element $\mathrm{id}_A \in \mathrm{Hom}_{\mathbb{H}}(A, A)$.

These data satisfy the following axioms.

(HC1) *(Hyperassociativity)* For all composable $f \in \mathrm{Hom}_{\mathbb{H}}(A, B)$, $g \in \mathrm{Hom}_{\mathbb{H}}(B, C)$, $h \in \mathrm{Hom}_{\mathbb{H}}(C, D)$,
$$\bigcup_{u \in (g \odot f)} (h \odot u) \;=\; \bigcup_{v \in (h \odot g)} (v \odot f) \;\subseteq\; \mathrm{Hom}_{\mathbb{H}}(A, D).$$

(HC2) *(Unit laws)* For all $f \in \mathrm{Hom}_{\mathbb{H}}(A, B)$,
$$\mathrm{id}_B \odot f = \{f\}, \qquad f \odot \mathrm{id}_A = \{f\}.$$

If each $g \odot f$ is always a singleton, then $\mathbb{H}$ is a *strict* hypercategory; in this case, hypercomposition is ordinary single-valued composition, and $\mathbb{H}$ is an ordinary category.

Definition 3.10.6 (n-SuperHyperClass)**.** Let S be a set and let $\mathcal{C}^{(1)} \subseteq \mathcal{P}(S)$ be a nonempty collection. (Optionally, equip $\mathcal{C}^{(1)}$ with a hyperoperation as in Definition 3.10.4.) An *n-SuperHyperClass over S* is a recursively defined hierarchy
$$\mathcal{C}^{(1)}, \mathcal{C}^{(2)}, \ldots, \mathcal{C}^{(n)}$$
such that for each $k \geq 2$ there exists a predicate φ_k on nonempty subcollections of $\mathcal{C}^{(k-1)}$ and
$$\mathcal{C}^{(k)} := \big\{\, \mathcal{A} \subseteq \mathcal{C}^{(k-1)} \;\big|\; \mathcal{A} \neq \varnothing \text{ and } \varphi_k(\mathcal{A}) \,\big\}.$$
Thus every element of $\mathcal{C}^{(k)}$ is itself a nonempty collection of elements of $\mathcal{C}^{(k-1)}$, so the hierarchy is genuinely *nested*.

Remark 3.10.7. If $\mathcal{C}^{(1)} \subseteq \mathcal{P}(S)$, then $\mathcal{C}^{(2)} \subseteq \mathcal{P}(\mathcal{C}^{(1)}) \subseteq \mathcal{P}\big(\mathcal{P}(S)\big) = \mathcal{P}^2(S)$, and similarly $\mathcal{C}^{(k)} \subseteq \mathcal{P}^k(S)$ for all k.

Example 3.10.8 (A 2-SuperHyperClass of pairwise-intersecting families)**.** Let $U = \{1, 2, 3\}$ and take

$$\mathcal{C}^{(1)} := \mathcal{P}(U) \setminus \{\varnothing\} = \big\{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\big\}.$$

Define a predicate φ_2 on nonempty subcollections $\mathcal{A} \subseteq \mathcal{C}^{(1)}$ by

$$\varphi_2(\mathcal{A}) \ :\Longleftrightarrow\ \big(\forall X, Y \in \mathcal{A}\big)\ (X \cap Y \neq \varnothing).$$

Then

$$\mathcal{C}^{(2)} := \big\{\, \mathcal{A} \subseteq \mathcal{C}^{(1)} \ \big|\ \mathcal{A} \neq \varnothing \text{ and } \varphi_2(\mathcal{A}) \,\big\}$$

is a concrete 2-SuperHyperClass. For instance,

$$\mathcal{A}_0 := \big\{\{1\}, \{1, 2\}, \{1, 3\}\big\} \in \mathcal{C}^{(2)}, \quad \text{since every pair of members intersects.}$$

Definition 3.10.9 (n-SuperHyperCategory)**.** Fix a base set S and an n-SuperHyperClass hierarchy $\mathcal{C}^{(1)}, \ldots, \mathcal{C}^{(n)}$ as in Definition 3.10.6. An *n-SuperHyperCategory* is a hypercategory

$$\mathbb{H}^{(n)} = \big(\mathrm{Ob}(\mathbb{H}^{(n)}), \mathrm{Hom}, \odot, \mathrm{id}\big)$$

in the sense of Definition 3.10.5 such that:

(SHCat1) $\mathrm{Ob}(\mathbb{H}^{(n)}) \subseteq \mathcal{C}^{(n)}$ (objects are n-level superhyperclasses);

(SHCat2) for all objects $\mathcal{A}, \mathcal{B} \in \mathrm{Ob}(\mathbb{H}^{(n)})$, the hom-set $\mathrm{Hom}(\mathcal{A}, \mathcal{B})$ is a *collection of $(n-1)$-level objects*, i.e.

$$\mathrm{Hom}(\mathcal{A}, \mathcal{B}) \subseteq \mathcal{C}^{(n-1)};$$

(SHCat3) hypercomposition and identities satisfy the hypercategory axioms (HC1)–(HC2).

In particular, this definition makes precise the recursive slogan: *objects live one level higher than morphisms.*

Example 3.10.10 (A 2-SuperHyperCategory induced by supports)**.** Let $U = \{1, 2, 3\}$ and let $\mathcal{C}^{(1)} = \mathcal{P}(U)$. Consider the 2-SuperHyperClass $\mathcal{C}^{(2)}$ obtained from Example 3.10.8 (by restricting to nonempty, pairwise-intersecting families; thus $\mathcal{C}^{(2)} \subseteq \mathcal{P}(\mathcal{C}^{(1)})$). Take two objects

$$\mathcal{A}_0 = \big\{\{1\}, \{1, 2\}, \{1, 3\}\big\}, \qquad \mathcal{A}_1 = \big\{\{2\}, \{1, 2\}, \{2, 3\}\big\},$$

and set

$$\mathrm{Ob}(\mathbb{H}^{(2)}) = \{\mathcal{A}_0, \mathcal{A}_1\} \subseteq \mathcal{C}^{(2)}.$$

Support. For $\mathcal{A} \in \mathrm{Ob}(\mathbb{H}^{(2)})$, define its *support*

$$\mathrm{supp}(\mathcal{A}) := \bigcup_{X \in \mathcal{A}} X \subseteq U.$$

(Here $\mathrm{supp}(\mathcal{A}_0) = \{1, 2, 3\} = U = \mathrm{supp}(\mathcal{A}_1)$.)

Morphisms. For objects $\mathcal{A}, \mathcal{B}$ define

$$\mathrm{Hom}(\mathcal{A}, \mathcal{B}) := \mathcal{P}\big(\mathrm{supp}(\mathcal{A}) \cap \mathrm{supp}(\mathcal{B})\big) \subseteq \mathcal{P}(U) = \mathcal{C}^{(1)}.$$

Thus morphisms are $(n-1) = 1$ level objects (subsets of U), as required by (SHCat2).

Composition (strict). For $f \in \mathrm{Hom}(\mathcal{A}, \mathcal{B})$ and $g \in \mathrm{Hom}(\mathcal{B}, \mathcal{C})$ define

$$g \odot f := \{\, f \cap g \,\} \in \mathcal{P}\big(\mathrm{Hom}(\mathcal{A}, \mathcal{C})\big) \setminus \{\varnothing\}.$$

This is well-defined because $f \subseteq \mathrm{supp}(\mathcal{A}) \cap \mathrm{supp}(\mathcal{B})$ and $g \subseteq \mathrm{supp}(\mathcal{B}) \cap \mathrm{supp}(\mathcal{C})$, hence

$$f \cap g \subseteq \mathrm{supp}(\mathcal{A}) \cap \mathrm{supp}(\mathcal{C}),$$

so $f \cap g \in \mathrm{Hom}(\mathcal{A}, \mathcal{C})$.

Identities. Define

$$\mathrm{id}_{\mathcal{A}} := \mathrm{supp}(\mathcal{A}) \in \mathrm{Hom}(\mathcal{A}, \mathcal{A}).$$

Then for any $f \in \mathrm{Hom}(\mathcal{A}, \mathcal{B})$ we have

$$\mathrm{id}_{\mathcal{B}} \odot f = \{f \cap \mathrm{supp}(\mathcal{B})\} = \{f\}, \qquad f \odot \mathrm{id}_{\mathcal{A}} = \{\mathrm{supp}(\mathcal{A}) \cap f\} = \{f\},$$

so the unit laws (HC2) hold. Hyperassociativity (HC1) follows from associativity of $\cap$.

Hence $\mathbb{H}^{(2)}$ is a concrete 2-SuperHyperCategory in the precise sense of Definition 3.10.9.

3.11 Hypermagma and Superhypermagma

A *magma* is a nonempty set equipped with a binary operation, without any further requirements [142–144]. By contrast, a *group* is a magma whose operation is associative, which possesses an identity element, and in which every element has an inverse. The precise definition of a magma is given below.

Definition 3.11.1 (Magma)**.** [142–144] A *magma* is an ordered pair $(M, \cdot)$ where:

1. M is a nonempty set.

2. $\cdot \colon M \times M \to M$ is a binary operation on M.

That is, for every $a, b \in M$, the product $a \cdot b$ is a uniquely defined element of M.

Definition 3.11.2 (Group)**.** (cf. [145–147]) A *group* is a magma $(G, \cdot)$ that satisfies the following three axioms:

1. **Associativity:** For all $a, b, c \in G$,

$$(a \cdot b) \cdot c = a \cdot (b \cdot c).$$

2. **Identity:** There exists an element $e \in G$ (called the identity) such that for all $a \in G$,
$$e \cdot a = a \cdot e = a.$$

3. **Inverses:** For each $a \in G$, there exists an element $a^{-1} \in G$ (called the inverse of a) such that
$$a \cdot a^{-1} = a^{-1} \cdot a = e.$$

A hypermagma is a set with a hyperoperation mapping each pair to a subset, generalizing binary operations to set operations. The definition of a HyperMagma is described as follows [148–151].

Definition 3.11.3 (Hypermagma)**.** [151] A *hypermagma* is a pair $(H, \star)$ where:

1. H is a nonempty set.

2. $\star\colon H \times H \to \mathcal{P}(H)$ is a *hyperoperation*; that is, for each ordered pair $(x, y) \in H \times H$, the value
$$x \star y \subseteq H$$
is a (possibly empty) subset of H.

Often one requires that the hyperoperation be *total*, meaning
$$x \star y \neq \varnothing \quad \text{for all } x, y \in H.$$
This generalizes the notion of a binary operation by allowing the "product" of two elements to be a set of elements rather than a single element.

Example 3.11.4 (A finite total hypermagma)**.** Let $H = \{a, b\}$. Define a (total) hyperoperation $\star : H \times H \to \mathcal{P}_*(H)$ by
$$a \star a = \{a\}, \qquad a \star b = \{a, b\}, \qquad b \star a = \{b\}, \qquad b \star b = \{a\}.$$
Each value is a nonempty subset of H, hence $\star$ is total. Therefore $(H, \star)$ is a (total) hypermagma in the sense of the definition.

A superhypermagma generalizes hypermagmas by mapping each element pair to a nonempty subset in the nth powerset, and ensures closure [59]. The definition of a SuperHyperMagma is described as follows.

Definition 3.11.5 (SuperHyperMagma)**.** [59] Let H be a nonempty set and let $n \geq 1$ be a fixed positive integer. A *superhyperoperation* on H is a function
$$\circ\colon H \times H \to \mathcal{P}_n(H),$$
where $\mathcal{P}_n(H)$ denotes the n-th powerset of H (defined recursively by $\mathcal{P}_1(H) = \mathcal{P}(H)$ and $\mathcal{P}_{k+1}(H) = \mathcal{P}(\mathcal{P}_k(H))$ for $k \geq 1$). A pair $(H, \circ)$ is called a *superhyperMagma*. Notice that when $n = 1$ we recover the definition of a hypermagma. In this way, the notion of superhyperMagma generalizes that of a hypermagma.

Example 3.11.6 (A level-2 superhypermagma induced from a hypermagma)**.** Let $H = \{a, b\}$ and let $\star$ be the hyperoperation from Example 3.11.4. Fix $n = 2$. Recall that $\mathcal{P}_2(H) = \mathcal{P}(\mathcal{P}(H))$, so an element of $\mathcal{P}_2(H)$ is a set whose members are subsets of H.

Define a superhyperoperation $\circ : H \times H \to \mathcal{P}_2(H)$ by the singleton-lift

$$x \circ y \;:=\; \big\{\, A \subseteq H \mid A \in x \star y \,\big\}^{\uparrow} \;=\; \big\{\, \{u\} \;\big|\; u \in x \star y \,\big\},$$

i.e., we embed each output element $u \in x \star y$ as the singleton subset $\{u\} \in \mathcal{P}(H)$, and collect these singletons as a subset of $\mathcal{P}(H)$.

Concretely,

$$a \circ a = \big\{\{a\}\big\}, \qquad a \circ b = \big\{\{a\}, \{b\}\big\}, \qquad b \circ a = \big\{\{b\}\big\}, \qquad b \circ b = \big\{\{a\}\big\}.$$

Each value is a nonempty subset of $\mathcal{P}(H)$, hence an element of $\mathcal{P}_2(H)$. Therefore $(H, \circ)$ is a level-2 superhypermagma.

3.12 HyperMonoid and SuperHyperMonoid

A hypermonoid is a set with associative hyperoperation producing nonempty subsets and an identity element acting neutrally on both sides [152–154]. A superhyper monoid is a subset of an iterated powerset with associative operation and a two sided identity element present [155].

Definition 3.12.1 (Hypermonoid)**.** A *hypermonoid* is a triple $(M, \odot, e)$ where M is a nonempty set, $\odot$ is a hyperoperation on M, and $e \in M$ (identity) satisfies:

(i) **(Associativity)** For all $x, y, z \in M$,

$$(x \odot y) \odot z \;=\; x \odot (y \odot z),$$

where the equality is between subsets of M using the subset extension above.

(ii) **(Identity)** For all $x \in M$,

$$x \in e \odot x \qquad \text{and} \qquad x \in x \odot e.$$

If the stronger conditions $e \odot x = \{x\} = x \odot e$ hold for all x, we call e a *strong identity.*

Definition 3.12.2 (Superhyper monoid)**.** [155] Fix a nonempty set S and an integer $n \geq 1$. Let $\Omega \subseteq \mathcal{P}^n(S)$ be nonempty. A *superhyper monoid* is a pair (Ω, Θ) where

$$\Theta : \Omega \times \Omega \longrightarrow \Omega$$

is a binary operation such that:

(i) **(Closure)** $x \Theta y \in \Omega$ for all $x, y \in \Omega$.

(ii) **(Associativity)** $(x\Theta y)\Theta z = x\Theta(y\Theta z)$ for all $x, y, z \in \Omega$.

(iii) **(Identity)** There exists $\theta \in \Omega$ such that $x\Theta\theta = \theta\Theta x = x$ for all $x \in \Omega$.

Example 3.12.3 (A hypermonoid and a superhyper monoid)**. (1) Hypermonoid.** Let $M = \{0, 1\}$ and define a total hyperoperation $\odot : M \times M \to \mathcal{P}_*(M)$ by

$$0 \odot 0 = \{0\}, \qquad 0 \odot 1 = \{1\}, \qquad 1 \odot 0 = \{1\}, \qquad 1 \odot 1 = \{1\}.$$

Let $e := 0$. Then e is a *strong identity*:

$$e \odot x = 0 \odot x = \{x\}, \qquad x \odot e = x \odot 0 = \{x\} \qquad (x \in M).$$

Moreover, associativity holds because $\odot$ is actually single-valued (each output is a singleton), so $(x \odot y) \odot z = \{(x \cdot y) \cdot z\} = \{x \cdot (y \cdot z)\} = x \odot (y \odot z)$, where $\cdot$ denotes the usual Boolean "and" on $\{0, 1\}$. Hence $(M, \odot, e)$ is a hypermonoid.

(2) Superhyper monoid. Let the same base set be $S = \{0, 1\}$ and take $n = 1$. Then $\mathcal{P}^1(S) = \mathcal{P}(S)$. Let

$$\Omega := \mathcal{P}(S) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}.$$

Define a binary operation $\Theta : \Omega \times \Omega \to \Omega$ by set union:

$$X\Theta Y := X \cup Y \qquad (X, Y \in \Omega).$$

Then Θ is closed on Ω, associative, and has identity $\theta = \emptyset$:

$$(X\Theta Y)\Theta Z = (X \cup Y) \cup Z = X \cup (Y \cup Z) = X\Theta(Y\Theta Z), \qquad X\Theta\emptyset = \emptyset\Theta X = X.$$

Therefore (Ω, Θ) is a superhyper monoid (of level $n = 1$).

3.13 HyperIdeal and SuperHyperIdeal

A hyperideal is a nonempty subset closed under hyperproduct, absorbing all left and right products with arbitrary monoid elements universally [155]. A superhyperideal is a subset of a superhyper monoid, closed under Theta, absorbing Theta-products with all elements on both sides [155].

Definition 3.13.1 (Hyperideal of a hypermonoid)**.** [155] Let $(M, \odot, e)$ be a hypermonoid. A nonempty subset $I \subseteq M$ is a *(two-sided) hyperideal* if

$$I \odot I \subseteq I, \qquad M \odot I \subseteq I, \qquad I \odot M \subseteq I,$$

where $A \odot B$ is the subset product from the induced extension. (Left/right hyperideals are obtained by dropping one of the last two inclusions.)

Definition 3.13.2 (Superhyper ideal of a superhyper monoid)**.** [155] Let (Ω, Θ) be a superhyper monoid. A subset $I \subseteq \Omega$ is a *superhyper ideal* if:

(i) For all $x, y \in I$, one has $x\Theta y \in I$.

(ii) For all $x \in I$ and $y \in \Omega$, one has $x\Theta y \in I$ and $y\Theta x \in I$.

Example 3.13.3 (A hyperideal and a superhyper ideal)**. (1) Hyperideal of a hypermonoid.** Let $M = \{0, 1\}$ with the hypermonoid $(M, \odot, e)$ from Example 3.12.3:

$$0 \odot 0 = \{0\}, \quad 0 \odot 1 = \{1\}, \quad 1 \odot 0 = \{1\}, \quad 1 \odot 1 = \{1\}, \qquad e = 0.$$

Consider the nonempty subset $I := \{1\} \subseteq M$. Using the induced subset product $A \odot B = \bigcup_{a\in A,\, b\in B}(a \odot b)$, we compute:

$$I \odot I = \{1\} \odot \{1\} = 1 \odot 1 = \{1\} \subseteq I,$$
$$M \odot I = \{0, 1\} \odot \{1\} = (0 \odot 1) \cup (1 \odot 1) = \{1\} \cup \{1\} = \{1\} \subseteq I,$$
$$I \odot M = \{1\} \odot \{0, 1\} = (1 \odot 0) \cup (1 \odot 1) = \{1\} \cup \{1\} = \{1\} \subseteq I.$$

Hence $I = \{1\}$ is a (two-sided) hyperideal of the hypermonoid $(M, \odot, e)$.

(2) Superhyper ideal of a superhyper monoid. Let $S = \{0, 1\}$, $n = 1$, $\Omega = \mathcal{P}(S) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$, and let Θ be set union:

$$X\Theta Y := X \cup Y \qquad (X, Y \in \Omega).$$

Define

$$I := \{\, \{1\},\ \{0, 1\} \,\} \subseteq \Omega,$$

the family of all subsets of S that contain 1. Then:

- If $x, y \in I$, then $x \cup y$ still contains 1, so $x\Theta y = x \cup y \in I$.
- If $x \in I$ and $y \in \Omega$, then $x \cup y$ contains 1, hence $x\Theta y \in I$ and also $y\Theta x = y \cup x \in I$.

Therefore I is a superhyper ideal of the superhyper monoid (Ω, Θ).

3.14 Semihypergroups and SemiSuperHyperGroups

A semigroup is a set with an associative binary operation; closure holds, but identity and inverses are not required [156–158]. A semihypergroup is a set where an associative hyperoperation maps every pair of elements to a nonempty subset therein always [159–163]. A SemiSuperHyperGroup uses a superhyperoperation into an iterated powerset, extending hyperproducts, and satisfying strong associativity at higher nesting levels uniformly.

Definition 3.14.1 (Hyperoperation and induced subset product)**.** Let H be a nonempty set. A *(total) hyperoperation* on H is a map

$$\circ \ : \ H \times H \longrightarrow \mathcal{P}_*(H) := \mathcal{P}(H) \setminus \{\emptyset\}.$$

Given nonempty subsets $A, B \in \mathcal{P}_*(H)$, its canonical extension to subsets is

$$A \circ B \ := \bigcup_{a\in A} \bigcup_{b\in B} (a \circ b) \ \in \ \mathcal{P}_*(H).$$

(Equivalently, for $x \in H$ one sets $A \circ x := A \circ \{x\}$ and $x \circ B := \{x\} \circ B$.)

Definition 3.14.2 (Semihypergroup)**.** A *semihypergroup* is a pair $(H, \circ)$ where H is a nonempty set and $\circ : H \times H \to \mathcal{P}_*(H)$ is a hyperoperation such that, for all $x, y, z \in H$,

$$(x \circ y) \circ z \;=\; x \circ (y \circ z) \qquad \text{(equality of subsets of } H\text{)},$$

where the products with $(x \circ y)$ and $(y \circ z)$ are interpreted via the induced subset product. Equivalently,

$$\bigcup_{u \in x \circ y} (u \circ z) \;=\; \bigcup_{v \in y \circ z} (x \circ v).$$

Example 3.14.3 (A semihypergroup from a semigroup)**.** Let $(S, \cdot)$ be any semigroup (associative binary operation). Define a hyperoperation on S by

$$x \circ y \;:=\; \{\, x \cdot y \,\} \qquad (x, y \in S).$$

Then $(S, \circ)$ is a semihypergroup. Indeed, for all $x, y, z \in S$,

$$(x \circ y) \circ z = \{x \cdot y\} \circ z = \{(x \cdot y) \cdot z\}, \qquad x \circ (y \circ z) = x \circ \{y \cdot z\} = \{x \cdot (y \cdot z)\},$$

and these are equal because $\cdot$ is associative.

Real-world interpretation. Take S as the set of strings over an alphabet and $\cdot$ as concatenation. Then $x \circ y$ returns the (unique) concatenated string xy. Associativity models that parsing $(xy)z$ or $x(yz)$ yields the same final message.

Definition 3.14.4 (Level-n superhyperoperation and its extension)**.** Fix $n \geq 1$. A *(total) level-n superhyperoperation* on H is a map

$$\odot \;:\; H \times H \longrightarrow \mathcal{P}^n_*(H).$$

Using Flat_n, extend $\odot$ to act on level-n arguments by

$$X \odot Y \;:=\; \bigcup_{u \in \mathrm{Flat}_n(X)} \; \bigcup_{v \in \mathrm{Flat}_n(Y)} (u \odot v), \qquad X, Y \in \mathcal{P}^n_*(H),$$

and identify each $x \in H$ with $\iota_n(x) \in \mathcal{P}^n_*(H)$ when forming mixed products.

Definition 3.14.5 (SemiSuperhypergroup (level n))**.** A *(level-n) semiSuperhypergroup* is a pair $(H, \odot)$ where H is a nonempty set and $\odot : H \times H \to \mathcal{P}^n_*(H)$ is a level-n superhyperoperation satisfying *strong superhyperassociativity*: for all $x, y, z \in H$,

$$x \odot (y \odot z) \;=\; (x \odot y) \odot z \qquad \text{in } \mathcal{P}^n_*(H),$$

where products are interpreted via the above extension and the identification $x \simeq \iota_n(x)$.

Remark 3.14.6. When $n = 1$, we have $\mathcal{P}^1_*(H) = \mathcal{P}^*(H)$, Flat_1 is the identity on $\mathcal{P}^*(H)$, and Definition 3.14.5 reduces to the usual semihypergroup associativity in Definition 3.14.2.

Example 3.14.7 (A semiSuperhypergroup via n-nested singleton embedding)**.** Fix $n \geq 1$ and let $(S, \cdot)$ be a semigroup. Let $\iota_n : S \to \mathcal{P}_*^n(S)$ be the iterated singleton embedding

$$\iota_1(x) = \{x\}, \qquad \iota_{n+1}(x) = \{\iota_n(x)\}.$$

Define a level-n superhyperoperation $\odot : S \times S \to \mathcal{P}_*^n(S)$ by

$$x \odot y \ := \ \iota_n(x \cdot y) \qquad (x, y \in S).$$

Extend $\odot$ to level-n arguments using flattening. Then $(S, \odot)$ is a level-n semiSuperhypergroup, i.e.,

$$x \odot (y \odot z) \ = \ (x \odot y) \odot z \qquad \text{in } \mathcal{P}_*^n(S) \quad (x, y, z \in S).$$

Reason: by construction, both sides evaluate to $\iota_n(x \cdot (y \cdot z))$ and $\iota_n((x \cdot y) \cdot z)$ respectively, and these coincide by associativity of $\cdot$.

Let S be the set of software build artifacts and $\cdot$ be an associative "compose" operator (link $\to$ package $\to$ container). The output is stored as an n-layer nested bundle (e.g., artifact $\subset$ module $\subset$ release), while strong associativity guarantees that grouping intermediate build steps does not change the final bundled result.

3.15 Quasi-Hypergroups and Quasi-SuperHyperGroup

A quasi-hypergroup is a set with a hyperoperation whose left and right translations reproduce the whole set, but associativity is not required in general cases [164]. A quasi-superhypergroup generalizes this: the product of two elements yields an iterated set, and flattened left/right products still cover all base elements at every level.

Definition 3.15.1 (Hypergroupoid and induced subset product)**.** Let $H \neq \varnothing$. Write $\mathcal{P}_*(H) := \mathcal{P}(H) \setminus \{\varnothing\}$. A *(binary) hyperoperation* on H is a map

$$\circ \ : \ H \times H \longrightarrow \mathcal{P}_*(H).$$

The pair $(H, \circ)$ is called a *hypergroupoid* (cf. [56, 165, 166]). For nonempty subsets $A, B \subseteq H$, extend $\circ$ to subsets by

$$A \circ B \ := \ \bigcup_{a \in A} \bigcup_{b \in B} (a \circ b) \ \subseteq \ H, \qquad a \circ B := \{a\} \circ B, \quad A \circ b := A \circ \{b\}.$$

Definition 3.15.2 (Quasi-hypergroup)**.** A hypergroupoid $(H, \circ)$ is called a *quasi-hypergroup* if it satisfies the *reproduction axiom*:

$$a \circ H \ = \ H \ = \ H \circ a \qquad \text{for all } a \in H.$$

Remark 3.15.3 (Hypergroup as the associative case)**.** A quasi-hypergroup $(H, \circ)$ is called a *hypergroup* if, in addition, $\circ$ is associative in the canonical subset sense:

$$(a \circ b) \circ c \ = \ a \circ (b \circ c) \qquad \text{for all } a, b, c \in H,$$

where the products are computed using the subset extension above.

Example 3.15.4 (A simple quasi-hypergroup: "left-or-right choice")**.** Let H be any nonempty set with at least two elements, and define a binary hyperoperation

$$\circ : \ H \times H \longrightarrow \mathcal{P}_*(H), \qquad a \circ b \ := \ \{a, b\} \qquad (a, b \in H).$$

Then $(H, \circ)$ is a quasi-hypergroup. Indeed, for any fixed $a \in H$,

$$a \circ H \ = \ \bigcup_{x \in H} (a \circ x) \ = \ \bigcup_{x \in H} \{a, x\} \ = \ H,$$

because every $y \in H$ appears in the term $\{a, y\} = a \circ y$. Similarly,

$$H \circ a \ = \ \bigcup_{x \in H} (x \circ a) \ = \ \bigcup_{x \in H} \{x, a\} \ = \ H.$$

Hence the reproduction axiom holds.

Real-world interpretation. Think of H as a set of available options (e.g., two possible routes, two backup servers, two suppliers). Combining a and b returns the admissible set "choose either a or b," representing a nondeterministic fallback rule.

Definition 3.15.5 (Level-n superhyperoperation and induced mixed products)**.** Fix an integer $n \geq 1$. A *level-n superhyperoperation* on H is a map

$$\circ \ : \ H \times H \longrightarrow \mathcal{P}_*^{\,n}(H).$$

Extend $\circ$ to allow a set of right (or left) inputs by

$$a \circ B \ := \ \bigcup_{b \in B} (a \circ b) \ \in \ \mathcal{P}_*^{\,n}(H), \qquad B \circ a \ := \ \bigcup_{b \in B} (b \circ a) \ \in \ \mathcal{P}_*^{\,n}(H)$$

for every nonempty $B \subseteq H$. More generally, for $X \in \mathcal{P}_*^{\,n}(H)$ and $a \in H$, define

$$X \circ a \ := \ \bigcup_{u \in \mathrm{Flat}_n(X)} (u \circ a), \qquad a \circ X \ := \ \bigcup_{u \in \mathrm{Flat}_n(X)} (a \circ u),$$

and for $X, Y \in \mathcal{P}_*^{\,n}(H)$ define the canonical level-n product

$$X \circ Y \ := \ \bigcup_{u \in \mathrm{Flat}_n(X)} \ \bigcup_{v \in \mathrm{Flat}_n(Y)} (u \circ v) \ \in \ \mathcal{P}_*^{\,n}(H).$$

Definition 3.15.6 (Quasi-superhypergroup of level n)**.** Fix $n \geq 1$. A pair $(H, \circ)$, where $\circ : H \times H \to \mathcal{P}_*^{\,n}(H)$, is called a *quasi-superhypergroup (of level n)* if it satisfies the reproduction axiom at base level:

$$\mathrm{Flat}_n(a \circ H) \ = \ H \ = \ \mathrm{Flat}_n(H \circ a) \qquad \text{for all } a \in H.$$

Equivalently, for every $a \in H$ and every $x \in H$, there exist $b, c \in H$ such that

$$x \in \mathrm{Flat}_n(a \circ b) \quad \text{and} \quad x \in \mathrm{Flat}_n(c \circ a).$$

Remark 3.15.7 (Associativity leads to superhypergroups)**.** A quasi-superhypergroup $(H, \circ)$ is called a *(level-n) superhypergroup* if, moreover, $\circ$ is associative with respect to the canonical level-n extension:

$$(a \circ b) \circ c \ = \ a \circ (b \circ c) \qquad \text{in } \mathcal{P}_*^{\,n}(H) \text{ for all } a, b, c \in H,$$

where the mixed products are interpreted via Flat_n as above.

Example 3.15.8 (A quasi-superhypergroup of level n: nested "choice")**.** Fix an integer $n \geq 1$ and let H be any nonempty set with $|H| \geq 2$. Let $\iota_n : H \to \mathcal{P}_*^n(H)$ be the iterated singleton embedding

$$\iota_1(x) = \{x\}, \qquad \iota_{n+1}(x) = \{\iota_n(x)\}.$$

Define the level-n superhyperoperation

$$\circ : \ H \times H \longrightarrow \mathcal{P}_*^n(H), \qquad a \circ b \ := \ \iota_n(a) \ \cup \ \iota_n(b) \qquad (a, b \in H),$$

where the union is taken in $\mathcal{P}_*^n(H)$ (so the result is an n-nested nonempty set whose flattening contains $\{a, b\}$).

Then $(H, \circ)$ is a quasi-superhypergroup of level n. Indeed, for any fixed $a \in H$,

$$\mathrm{Flat}_n(a \circ H) \ = \ \mathrm{Flat}_n\Big(\bigcup_{x \in H} (a \circ x) \Big) \ = \ \bigcup_{x \in H} \mathrm{Flat}_n\big(\iota_n(a) \cup \iota_n(x)\big) \ = \ \bigcup_{x \in H} \{a, x\} \ = \ H,$$

and similarly $\mathrm{Flat}_n(H \circ a) = H$. Thus the reproduction axiom holds at the base level after flattening.

Take H as a set of competing hypotheses in decision-making (diagnoses, threat labels, or candidate models). The nested output $a \circ b$ stores the admissible alternatives at multiple abstraction layers (raw candidates $\to$ grouped candidates $\to$ audited candidates), while flattening recovers the underlying set of possible choices.

3.16 EL-hyperstructures and EL-Superhyperstructures

EL-hyperstructures arise from quasi-ordered semigroups: define hyperproduct $a * b$ as the principal end $[a \cdot b)_{\leq}$, ensuring associativity and capturing order-generated multi-valued composition [167–172]. EL-Superhyperstructures lift the Ends-lemma idea to iterated powersets, producing nested hyperproducts as upper-sets under a compatible quasi-order across layers uniformly.

Notation 3.16.1 (Principal up-set)**.** *A* quasi-order *on a set H is a relation $\leq$ that is reflexive and transitive. A* quasi-ordered semigroup *is a triple $(H, \cdot, \leq)$ where $(H, \cdot)$ is a semigroup and $\leq$ is a quasi-order (typically assumed compatible/monotone, but the EL-construction itself only needs $\leq$ to define principal up-sets).*

For $a \in H$, the principal up-set *generated by a is*

$$[a)_{\leq} \ := \ \{\, x \in H \mid a \leq x \,\}.$$

More generally, for any quasi-ordered set $(X, \preceq)$ and $x \in X$, write

$$[x)_{\preceq} \ := \ \{\, y \in X \mid x \preceq y \,\}.$$

Definition 3.16.2 (EL-hyperstructure (Ends-lemma based hyperstructure))**.** Let $(H, \cdot, \leq)$ be a quasi-ordered semigroup and let $\mathcal{P}_*(H)$ denote the set of all nonempty subsets of H. Define a binary hyperoperation $* : H \times H \to \mathcal{P}_*(H)$ by

$$a * b \ := \ [a \cdot b)_{\leq} \ = \ \{\, x \in H \mid a \cdot b \leq x \,\}.$$

The hyperstructure $(H, *)$ is called the *EL-hyperstructure associated to* $(H, \cdot, \leq)$. (For ordered/quasi-ordered semigroups this construction is known to yield an associative semihypergroup.)

Example 3.16.3 (EL-hyperstructure from time tasks: "at least" semantics)**.** Let $H = \mathbb{N}$ represent *nonnegative integer time units* (e.g., minutes). Equip H with the usual addition $\cdot := +$ and the usual order $\leq$. Then $(H, +, \leq)$ is a quasi-ordered semigroup.

The associated EL-hyperoperation $* : H \times H \to \mathcal{P}_*(H)$ is

$$a * b = [a + b)_{\leq} = \{\, x \in \mathbb{N} \mid a + b \leq x \,\}.$$

Hence $(\mathbb{N}, *)$ is an EL-hyperstructure.

If a and b are the estimated durations of two consecutive tasks, then $a + b$ is the *minimum* total time. The hyperproduct $a * b$ returns all *admissible* realized total times $x \geq a + b$ (e.g., delays, overhead, waiting time), capturing a conservative "at least" scheduling semantics.

Fix $m \in \mathbb{N}_0$ and define the iterated powersets

$$\mathcal{P}^0(H) := H, \qquad \mathcal{P}^{m+1}(H) := \mathcal{P}\big(\mathcal{P}^m(H)\big).$$

We now lift both the quasi-order and the product to level m.

Notation 3.16.4 (Lifted quasi-orders)**.** *Define recursively relations $\preceq_m$ on $\mathcal{P}^m(H)$ by*

$$\preceq_0 := \leq, \qquad A \preceq_{m+1} B \iff \big(\forall a \in A\, \exists b \in B :\ a \preceq_m b\big), \qquad A, B \in \mathcal{P}^{m+1}(H).$$

Each $\preceq_m$ is a quasi-order (the standard "Hoare" lifting of a preorder).

Notation 3.16.5 (Lifted products)**.** *Define recursively operations $\odot_m$ on $\mathcal{P}^m(H)$ by*

$$\odot_0 := \cdot, \qquad A \odot_{m+1} B := \{\, a \odot_m b \mid a \in A,\ b \in B \,\}, \qquad A, B \in \mathcal{P}^{m+1}(H).$$

Thus $\odot_m : \mathcal{P}^m(H) \times \mathcal{P}^m(H) \to \mathcal{P}^m(H)$ is well-typed for every m.

Definition 3.16.6 (EL-SuperHyperStructure of level m)**.** Let $(H, \cdot, \leq)$ be a quasi-ordered semigroup and fix $m \in \mathbb{N}_0$. Using the lifted quasi-order $\preceq_m$ and lifted product $\odot_m$ defined above, define

$$\star_m :\ \mathcal{P}^m(H) \times \mathcal{P}^m(H) \longrightarrow \mathcal{P}^{m+1}(H)$$

by the *principal up-set at level m*:

$$A \star_m B := [A \odot_m B)_{\preceq_m} = \big\{\, C \in \mathcal{P}^m(H) \;\big|\; A \odot_m B \preceq_m C \,\big\}.$$

Then $(\mathcal{P}^m(H), \star_m)$ is a (binary) *superhyperstructure* of profile $(m, m+1, 2)$ over the base set H, called the *EL-SuperHyperStructure (level m) associated to* $(H, \cdot, \leq)$.

Example 3.16.7 (EL-SuperHyperStructure from sets of tasks (level $m = 1$))**.** Continue with $H = \mathbb{N}$, $\cdot := +$, and $\leq$. Fix $m = 1$, so $\mathcal{P}^1(H) = \mathcal{P}(H)$. For nonempty $A, B \subseteq \mathbb{N}$, the lifted product is the Minkowski sum

$$A \odot_1 B = \{\, a + b \mid a \in A,\ b \in B \,\},$$

and the lifted quasi-order $\preceq_1$ is the Hoare lifting:

$$A \preceq_1 C \quad \Longleftrightarrow \quad \forall s \in A \; \exists t \in C \text{ such that } s \leq t.$$

The EL-superhyperoperation $\star_1 : \mathcal{P}(H) \times \mathcal{P}(H) \to \mathcal{P}^2(H)$ is

$$A \star_1 B = [A \odot_1 B)_{\preceq_1} = \big\{ C \in \mathcal{P}(H) \;\big|\; A \odot_1 B \preceq_1 C \big\}.$$

Let A be a set of *possible durations* for task 1 (e.g., best/typical/worst cases) and B a set of possible durations for task 2. Then $A \odot_1 B$ is the set of all possible *minimum* total durations. The value $A \star_1 B$ is a *set of sets* of durations: it consists of all nonempty sets C that *dominate* $A \odot_1 B$ in the Hoare sense, i.e., for every minimum scenario $s \in A \odot_1 B$, the set C contains some feasible realized time $t \geq s$. Thus $A \star_1 B$ models admissible *risk buffers / slack policies* that cover all combined scenarios.

3.17 Compact Hypergroup and Compact Superhypergroup

Compact hypergroups are hypergroups whose underlying space is compact (typically compact Hausdorff) and whose multivalued convolution/product is continuous in the appropriate hyperspace (e.g., Vietoris) topology, with identity, associativity, and involutive inverse axioms holding [173–175]. Compact superhypergroups are higher-level analogues built on compact spaces in which the product maps tuples of (super)elements into iterated hyperspaces (nested set-valued outputs), and the corresponding "super" versions of continuity, identity, associativity, and inverse axioms are satisfied.

To speak about *compactness*, we equip the underlying set with a topology and require the (super)hyperoperation to be continuous as a set-valued map (via the Vietoris hyperspace topology).

Notation 3.17.1. *Let X be a compact Hausdorff space. Denote by $\mathcal{K}(X)$ the set of all* nonempty compact *subsets of X, endowed with the Vietoris topology, whose subbase consists of sets of the form*

$$\langle U \rangle^+ := \{K \in \mathcal{K}(X) \mid K \subseteq U\}, \qquad \langle U \rangle^- := \{K \in \mathcal{K}(X) \mid K \cap U \neq \varnothing\},$$

where $U \subseteq X$ is open. Define iterated hyperspaces recursively by

$$\mathcal{K}^0(X) := X, \qquad \mathcal{K}^{n+1}(X) := \mathcal{K}\big(\mathcal{K}^n(X)\big) \quad (n \geq 0).$$

(When X is finite with the discrete topology, every subset is compact, so $\mathcal{K}(X) = \mathcal{P}_*(X)$ and $\mathcal{K}^n(X) = \mathcal{P}_*^{\,n}(X)$.)

Recall that a (group-like) hypergroup is a triple $(H, *, e)$ with a binary hyperoperation $* : H \times H \to \mathcal{P}_*(H)$ satisfying associativity, identity, and inverses.

Definition 3.17.2 (Compact hypergroup)**.** [173–175] A *compact hypergroup* is a quadruple $(H, \tau, *, e)$ such that:

1. $(H, *, e)$ is a (group-like) hypergroup.

2. (H, τ) is a compact Hausdorff topological space.

3. The hyperoperation is *topologically continuous* in the hyperspace sense, i.e.,

$$* : \ (H \times H, \ \tau \times \tau) \longrightarrow \big(\mathcal{K}(H), \ \text{Vietoris}\big), \qquad (x, y) \longmapsto x * y,$$

is continuous, where each value $x * y$ is viewed as a nonempty compact subset of H.

Remark 3.17.3. Equivalently, Item (3) means that for every open $U \subseteq H$ the sets $\{(x, y) \mid x * y \subseteq U\}$ and $\{(x, y) \mid (x * y) \cap U \neq \varnothing\}$ are open in $H \times H$.

Example 3.17.4 (Compact hypergroup from planar rotations)**.** Let

$$H := \mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\},$$

equipped with the usual (subspace) topology τ, which is compact and Hausdorff. Let the identity be $e := 1 \in \mathbb{S}^1$ and define the hyperoperation

$$* : \ H \times H \longrightarrow \mathcal{K}(H), \qquad x * y \ := \ \{xy\} \qquad (x, y \in H),$$

where $\mathcal{K}(H)$ denotes the hyperspace of nonempty compact subsets of H endowed with the Vietoris topology. Then $(H, *, e)$ is a (degenerate) hypergroup: associativity, identity, and inverses hold because $*$ is singleton-valued and induced by the usual group law on $\mathbb{S}^1$. Moreover, the map $(x, y) \mapsto \{xy\}$ is continuous as a map

$$(H \times H, \tau \times \tau) \longrightarrow (\mathcal{K}(H), \text{Vietoris}),$$

so $(H, \tau, *, e)$ is a compact hypergroup.

Elements of $\mathbb{S}^1$ represent *planar orientations* (angles modulo 2π), e.g., a robot's heading or a signal's phase. The product xy corresponds to composing rotations (adding angles mod 2π), while $\{xy\}$ records the (set-valued) outcome in hyperspace form.

A level-n superhypergroup is a triple $(H, \circ, e)$ whose operation returns an n-nested nonempty set and satisfies strong superhyperassociativity, identity, and inverses via flattening.

Definition 3.17.5 (Compact superhypergroup)**.** Fix an integer $n \geq 1$. A *compact superhypergroup of level* n is a quadruple $(H, \tau, \circ, e)$ such that:

1. $(H, \circ, e)$ is a level-n superhypergroup (with the singleton embedding ι_n and flattening Flat_n).

2. (H, τ) is a compact Hausdorff space.

3. The superhyperoperation is continuous as a map into the *iterated hyperspace*:

$$\circ : \ (H \times H, \ \tau \times \tau) \longrightarrow \big(\mathcal{K}^n(H), \ \text{iterated Vietoris}\big), \qquad (x, y) \longmapsto x \circ y,$$

where each value $x \circ y$ is regarded as an element of $\mathcal{K}^n(H)$ (i.e., an n-nested nonempty compact set).

Remark 3.17.6. If H is finite (discrete), then $\mathcal{K}^n(H) = \mathcal{P}_*^n(H)$ and continuity is automatic. Hence every finite (super)hypergroup is compact in the above sense.

Example 3.17.7 (Compact superhypergroup (level n) from hierarchical rotation composition)**.** Fix an integer $n \geq 1$ and let (H, τ) be as in Example 3.17.4, i.e., $H = \mathbb{S}^1$ with its usual compact Hausdorff topology. Define the level-n superhyperoperation $\circ$ by the iterated singleton embedding

$$\iota_n : H \longrightarrow \mathcal{P}_*^n(H), \qquad \iota_1(x) = \{x\}, \;\; \iota_{n+1}(x) = \{\iota_n(x)\},$$

and set

$$\circ : H \times H \longrightarrow \mathcal{P}_*^n(H), \qquad x \circ y \;:=\; \iota_n(xy) \qquad (x, y \in H),$$

with identity element $e := 1$.

Let Flat_n denote the standard flattening (iterated union) map $\mathrm{Flat}_n : \mathcal{P}_*^n(H) \to \mathcal{P}_*(H)$. Then the flattened hyperoperation is

$$x \diamond y \;:=\; \mathrm{Flat}_n(x \circ y) = \{xy\},$$

which is exactly the compact-hypergroup operation from Example 3.17.4. Hence $(H, \circ, e)$ is a level-n superhypergroup (all superhypergroup axioms hold because they hold after flattening to the underlying group law). In addition, viewing $x \circ y = \iota_n(xy)$ as an element of the iterated hyperspace $\mathcal{K}^n(H)$, the map

$$\circ : (H \times H, \tau \times \tau) \longrightarrow (\mathcal{K}^n(H), \text{iterated Vietoris})$$

is continuous (it is the composition of the continuous multiplication $(x, y) \mapsto xy$ with the continuous iterated singleton embedding into $\mathcal{K}^n(H)$). Therefore $(H, \tau, \circ, e)$ is a compact superhypergroup of level n.

In multi-sensor fusion for orientation/phase, one may store a *hierarchical* representation of a state (e.g., raw estimate $\to$ filtered estimate $\to$ decision-level estimate). The nesting level n models the *layering* of representation, while $x \circ y = \iota_n(xy)$ encodes rotation composition consistently across levels.

3.18 Hyper BCI-algebras and Superhyper BCI-algebras

A BCI-algebra is an algebraic structure $(A, \to, 0)$ satisfying BCI axioms, generalizing implication-based logic; it induces a partial order and supports algebraic reasoning [176–179]. A *hyper BCI-algebra* is a set equipped with a binary hyperoperation and a distinguished constant 0 that satisfies BCI-type axioms formulated via the induced hyperorder and closure conditions [180–182]. A *superhyper BCI-algebra* is an iterated nonempty-powerset lifting of a hyper BCI-algebra, where the operation and the zero element are lifted recursively so that the same axioms hold at the higher level (cf. [183]).

Let $H \neq \varnothing$ be a set. Write $\mathcal{P}^*(H) := \{A \subseteq H \mid A \neq \varnothing\}$.

Definition 3.18.1 (Hyperoperation and subset extension)**.** A *binary hyperoperation* on H is a map

$$\circ : H \times H \longrightarrow \mathcal{P}^*(H).$$

It extends canonically to nonempty subsets by

$$A \circ B \;:=\; \bigcup_{a \in A,\; b \in B} (a \circ b), \qquad (A, B \in \mathcal{P}^*(H)).$$

Definition 3.18.2 (Hyperorder induced by a constant)**.** Fix $0 \in H$. Define a binary relation $\prec$ on H by

$$x \prec y \quad :\Longleftrightarrow \quad 0 \in x \circ y.$$

For $A, B \in \mathcal{P}^*(H)$, define

$$A \prec B \quad :\Longleftrightarrow \quad (\forall a \in A)\,(\exists b \in B)\ (a \prec b).$$

Definition 3.18.3 (Hyper BCI-algebra)**.** A triple $(H, \circ, 0)$ is called a *hyper BCI-algebra* if $\circ : H \times H \to \mathcal{P}^*(H)$ is a binary hyperoperation and the following axioms hold for all $x, y, z \in H$:

(B1) $(x \circ z) \circ (y \circ z) \ \prec\ x \circ y.$

(B2) $(x \circ y) \circ z \ =\ (x \circ z) \circ y.$

(B3) $x \prec x.$

(B4) $x \prec y$ and $y \prec x$ imply $x = y.$

(B5) $0 \circ (0 \circ x) \ \prec\ x$ whenever $x \neq 0.$

Example 3.18.4 (A concrete finite hyper BCI-algebra)**.** Let

$$H = \{0, 1, 2\}, \qquad 0 \in H$$

and define the (singleton-valued) hyperoperation $\circ : H \times H \to \mathcal{P}^*(H)$ by

$$x \circ y \ :=\ \big\{\max(0, x - y)\big\} \qquad (x, y \in H),$$

where $\mathcal{P}^*(H) = \{A \subseteq H :\ A \neq \varnothing\}$. Equivalently, the table of $\circ$ is

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{0\}$
2	$\{2\}$	$\{1\}$	$\{0\}$

Define $x \prec y \iff 0 \in x \circ y$ and extend $\prec$ to nonempty subsets $A, B \in \mathcal{P}^*(H)$ by

$$A \prec B \iff \forall a \in A\ \exists b \in B \text{ such that } a \prec b.$$

Then the triple $(H, \circ, 0)$ satisfies axioms (B1)–(B5) of Definition 3.18.3; hence it is a (nontrivial) hyper BCI-algebra.

Definition 3.18.5 (Lifted hyperoperation and lifted zero)**.** Let $(H, \circ, 0)$ be a hyper BCI-algebra. Define recursively a sequence of carriers, operations, and distinguished zeros:

$$H^{(n)} := \mathcal{P}^n_*(H), \qquad 0^{(0)} := 0, \qquad 0^{(n+1)} := \{0^{(n)}\}\ (n \geq 0).$$

Define $\circ^{(0)} := \circ$, and for $n \geq 0$ define the *lift* $\circ^{(n+1)}$ by

$$\circ^{(n+1)} :\ H^{(n+1)} \times H^{(n+1)} \longrightarrow \mathcal{P}^*\big(H^{(n+1)}\big), \qquad A \circ^{(n+1)} B \ :=\ \{\, X \circ^{(n)} Y \mid X \in A,\ Y \in B \,\}.$$

As usual, each $\circ^{(n)}$ extends to nonempty subsets of $H^{(n)}$ via unions:

$$\mathcal{A} \circ^{(n)} \mathcal{B} := \bigcup_{a \in \mathcal{A},\, b \in \mathcal{B}} (a \circ^{(n)} b), \qquad \mathcal{A}, \mathcal{B} \in \mathcal{P}^*\big(H^{(n)}\big).$$

Definition 3.18.6 (SuperHyper BCI-algebra of depth n)**.** Fix $n \in \mathbb{N}$. With notation as in Definition 3.18.5, the triple

$$\left(H^{(n)},\ \circ^{(n)},\ 0^{(n)}\right)$$

is called an *n-SuperHyper BCI-algebra* (or a *SuperHyper BCI-algebra of depth n*) if it satisfies the hyper BCI axioms (B1)–(B5) of Definition 3.18.3 with $H, \circ, 0$ replaced by $H^{(n)}, \circ^{(n)}, 0^{(n)}$.

Example 3.18.7 (A concrete SuperHyper BCI-algebra (any depth))**.** Let $H = \{0\}$ and define $\circ : H \times H \to \mathcal{P}^*(H)$ by

$$0 \circ 0 := \{0\}.$$

Then $(H, \circ, 0)$ is a hyper BCI-algebra (all axioms are immediate because there is only one element).

For any $n \in \mathbb{N}$, form the iterated carriers and lifted zero as in Definition 3.18.5:

$$H^{(n)} := \mathcal{P}_*^n(H), \qquad 0^{(0)} := 0, \qquad 0^{(n+1)} := \{0^{(n)}\}.$$

Since $H = \{0\}$, each $H^{(n)}$ is again a singleton, namely

$$H^{(n)} = \{\, 0^{(n)} \,\}.$$

Moreover the lifted hyperoperation $\circ^{(n)}$ is necessarily the singleton-valued rule

$$0^{(n)} \circ^{(n)} 0^{(n)} = \{\, 0^{(n)} \,\}.$$

Therefore the triple $\left(H^{(n)}, \circ^{(n)}, 0^{(n)}\right)$ satisfies (B1)–(B5) for every n, and hence it is an n-SuperHyper BCI-algebra (a SuperHyper BCI-algebra of depth n) in the sense of Definition 3.18.6.

3.19 UP-hyperalgebras and UP-Superhyperalgebras

A UP-hyperalgebra is a set H with constant 0 and hyperoperation $\circ : H \times H \to \mathcal{P}^*(H)$, satisfying UP axioms using relation $x \ll y \Leftrightarrow 0 \in x \circ y$, ensuring antisymmetry, zero-dominance, and closure-like behavior overall [184]. A UP-superhyperalgebra equips H and 0 with a level-n superhyperoperation $\odot : H \times H \to \mathcal{P}^{*n}(H)$; flattening by iterated union yields a UP-hyperalgebra, capturing nested uncertainty while retaining the axioms.

Definition 3.19.1 (UP-hyperalgebra)**.** [184] Let H be a nonempty set, let $0 \in H$, and let $\circ : H \times H \to \mathcal{P}_*(H)$ be a binary hyperoperation. For nonempty $A, B \subseteq H$, define the induced hyperproduct by

$$A \circ B := \bigcup_{a \in A,\ b \in B} (a \circ b), \qquad A \circ x := A \circ \{x\}, \qquad x \circ B := \{x\} \circ B.$$

Define a relation $\ll$ on H by

$$x \ll y \iff 0 \in x \circ y \qquad (x, y \in H),$$

and extend it to subsets by

$$A \ll B \iff \forall a \in A\ \exists b \in B \text{ such that } a \ll b \qquad (A, B \subseteq H).$$

Then $(H, \circ, 0)$ is called a *UP-hyperalgebra* if it satisfies:

(UPh-1) $\forall x, y, z \in H, \quad (y \circ z) \ll (x \circ y) \circ (x \circ z)$;

(UPh-2) $\forall x \in H, \quad x \in 0 \circ x$;

(UPh-3) $\forall x \in H, \quad x \ll 0$;

(UPh-4) $\forall x, y \in H, \quad (x \ll y \ \wedge \ y \ll x) \Rightarrow x = y$.

Example 3.19.2 (Real-world cybersecurity example of a UP-hyperalgebra)**.** Let H be a finite set of *security postures* (states) for an enterprise service, for example

$$H = \{\texttt{normal}, \ \texttt{hardened}, \ \texttt{isolated}, \ \texttt{maintenance}, \ \dots\},$$

and let $0 \in H$ be a distinguished *safe baseline posture* ("fail-safe"): the state that is guaranteed to block external impact (e.g. network isolation plus least privilege).

For two postures $x, y \in H$, define $x \circ y \subseteq H$ to be the set of all postures that can result from *applying the mitigation policy x under the contextual constraints encoded by y*. The set-valuedness models operational nondeterminism: different subsystems (firewall, IAM, service mesh, SIEM) may enforce compatible mitigations in different orders, producing multiple admissible resulting postures.

Interpret

$$x \ll y \quad \Longleftrightarrow \quad 0 \in x \circ y$$

as "under context y, applying x guarantees a path to the fail-safe baseline 0" (e.g. x triggers isolation if y indicates an ongoing incident). Then the axioms (UPh-1)–(UPh-4) have a natural safety meaning:

- (UPh-2) $x \in 0 \circ x$ says that starting from the fail-safe baseline and applying any posture x does not lose feasibility of x (baseline can safely transition toward x when allowed).
- (UPh-3) $x \ll 0$ says that in the strongest safe context (baseline), every posture admits a mitigation route back to baseline (recovery is always possible).
- (UPh-4) antisymmetry makes the safety preorder $\ll$ a partial order: two postures that mutually guarantee reduction to baseline under each other must in fact be the same posture.
- (UPh-1) expresses a *composition-stability* principle: if $y \circ z$ can be driven to baseline, then after applying a common "control" action x to both y and z, their combined enforcement $(x \circ y) \circ (x \circ z)$ still preserves the ability to reach baseline.

Hence $(H, \circ, 0)$ provides an abstract UP-hyperalgebra model for *nondeterministic mitigation composition* in incident response and security hardening.

Definition 3.19.3 (UP-superhyperalgebra of level n)**.** Fix an integer $n \geq 1$. Let H be a nonempty set, let $0 \in H$, and let

$$\odot : \ H \times H \longrightarrow \mathcal{P}_*^n(H)$$

be a binary *level-n superhyperoperation* (i.e., each $x \odot y$ is a nonempty n-nested collection of elements of H). Define the induced *flattened hyperoperation* $\circ : H \times H \to \mathcal{P}_*(H)$ by

$$x \circ y \ := \ \mu_n(x \odot y) \qquad (x, y \in H),$$

where $\mu_n : \mathcal{P}_*^n(H) \to \mathcal{P}_*(H)$ is the iterated-union flattening map above. We call $(H, \odot, 0)$ a *UP-superhyperalgebra of level n* if the flattened triple $(H, \circ, 0)$ is a UP-hyperalgebra in the sense of Definition 3.19.1.

In particular, when $n = 1$ one has $\mu_1 = \mathrm{id}$ and the notion reduces to an ordinary UP-hyperalgebra.

Example 3.19.4 (Real-world cybersecurity example of a UP-superhyperalgebra (hierarchical mitigation choices))**.** In modern security orchestration, mitigation is often performed in *layers*: organization-wide controls (zero-trust posture), then service-level controls (rate limiting, feature flags), then host-level controls (EDR quarantine), etc. Each layer may admit multiple valid choices, producing a *hierarchy of alternatives*.

Let H and the baseline $0 \in H$ be as in Example 3.19.2. Fix $n \geq 2$ and define a level-n superhyperoperation

$$x \odot y \in \mathcal{P}_*^n(H)$$

so that the outer levels encode *which mitigation layer* (or orchestration playbook family) is selected, while inner levels encode admissible subsystem-specific implementations within that layer. Thus $x \odot y$ is a nonempty n-nested family of possible resulting postures.

Flattening via μ_n,

$$x \circ y := \mu_n(x \odot y) \in \mathcal{P}_*(H),$$

collects all concrete postures reachable by *some* chain of hierarchical choices. Declaring $(H, \odot, 0)$ to be a UP-superhyperalgebra of level n precisely requires that this induced (flattened) mitigation semantics still satisfies (UPh-1)–(UPh-4). Conceptually, it means that even with multi-layer orchestration branching, the system remains *baseline-recoverable*, *order-consistent*, and *composition-stable* in the UP-hyperalgebra sense.

3.20 Boolean Hyperalgebras and Boolean Superhyperalgebras

Boolean algebras are distributive complemented lattices $(B, \vee, \wedge, ', 0, 1)$ modeling classical logic, with idempotent, associative, commutative operations and De Morgan laws [185–188]. A Boolean hyperalgebra generalizes Boolean algebra by allowing join to return a nonempty set of elements, while meet, complement, 0,1 satisfy Boolean-like axioms still formally [189–191]. A Boolean superhyperalgebra further lifts join to an iterated nonempty powerset, producing nested outputs; flattening these outputs yields a Boolean hyperalgebra structure on its carrier.

Notation 3.20.1. *Given a hyperoperation $\vee : A \times A \to \mathcal{P}(A)$, extend it to subsets $X, Y \subseteq A$ by*

$$X \vee Y \;:=\; \bigcup_{x \in X} \bigcup_{y \in Y} (x \vee y).$$

Given an ordinary binary operation $\wedge : A \times A \to A$, extend it to $A \times \mathcal{P}(A)$ and $\mathcal{P}(A) \times A$ by

$$a \wedge Y \;:=\; \{\, a \wedge y : y \in Y \,\}, \qquad X \wedge a \;:=\; \{\, x \wedge a : x \in X \,\}.$$

(Thus expressions such as $a \vee (b \vee c)$ and $a \wedge (b \vee c)$ are interpreted via these extensions.)

Definition 3.20.2 (Boolean hyperalgebra)**.** A *Boolean hyperalgebra* is a sextuple

$$\mathbf{A} = (A,\ \vee,\ \wedge,\ 0,\ 1,\ ')$$

where $A \neq \varnothing$, $\vee : A \times A \to \mathcal{P}(A)$ is a (set-valued) hyperoperation, $\wedge : A \times A \to A$ is an ordinary binary operation, $0, 1 \in A$ are distinguished constants, and $' : A \to A$ is a unary operation, such that for all $a, b, c \in A$:

(BH1) $a \in a \vee a$ and $a = a \wedge a$;

(BH2) $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$;

(BH3) $a \vee (b \vee c) = (a \vee b) \vee c$ and $a \wedge (b \wedge c) = (a \wedge b) \wedge c$;

(BH4) $a \in a \vee (a \wedge b)$ and $a = a \wedge (a \vee b)$;

(BH5) $\big(a \in a \vee b \text{ and } b \in a \vee b\big) \iff a \wedge b = a$;

(BH6) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$;

(BH7) $0 \vee a = \{a\}$, $0 \wedge a = 0$, $1 \in 1 \vee a$, and $1 \wedge a = a$;

(BH8) $a \vee a' = \{1\}$ and $a \wedge a' = 0$.

Example 3.20.3 (Real-world computer-science example of a Boolean hyperalgebra)**.** Let A be a finite set of *access-control policies* for a microservice API, where each element $a \in A$ denotes a concrete policy configuration (e.g. a JSON rule set). Fix two distinguished policies: 0 is the *deny-all* policy and 1 is the *allow-all* policy. Define a' to be the *complement* policy (swap allow/deny outcomes on every request pattern).

In policy management, "OR" is often *nondeterministic* because a deployment may choose one of several equivalent safe merges (e.g. due to different precedence rules, tie-breaking among overlapping rules, or multiple valid normal forms produced by compilers). Model this by a set-valued operation

$$a \vee b \subseteq A,$$

where $a \vee b$ is the set of all policy configurations obtainable by *sound* merge strategies for a and b (e.g. "prefer explicit denies", "prefer tenant overrides", "prefer most specific rule", etc.). In contrast, define

$$a \wedge b \in A$$

as a *deterministic* conjunction operator implemented by the policy engine, for instance the policy that permits a request iff both a and b permit it.

With these interpretations, the axioms (BH1)–(BH8) express familiar correctness constraints: idempotence means merging a policy with itself yields itself among the valid outcomes; commutativity/associativity correspond to merge-order independence up to admissible outcomes; 0 and 1 behave as deny/allow identities; and $a \vee a' = \{1\}$ together with $a \wedge a' = 0$ formalize that combining a policy with its complement yields the universally permissive policy under "OR", while their conjunction yields deny-all. Thus the sextuple $(A, \vee, \wedge, 0, 1, ')$ can serve as an abstract Boolean hyperalgebra capturing *nondeterministic policy composition* in real access-control pipelines.

Definition 3.20.4 (Boolean n-SuperHyperalgebra)**.** Fix $n \geq 1$. A *Boolean n-SuperHyperalgebra* is a sextuple

$$\mathbf{A}^{(n)} = (A,\ \vee^{(n)},\ \wedge,\ 0,\ 1,\ ')$$

where $A \neq \varnothing$, $\vee^{(n)} : A \times A \to \mathcal{P}^n_*(A)$ is a superhyperoperation, $\wedge : A \times A \to A$ is an ordinary operation, and $0, 1 \in A$, $' : A \to A$, such that the derived (ordinary) hyperoperation

$$a\ \vee\ b\ :=\ \flat_n\big(a \vee^{(n)} b\big)\ \in\ \mathcal{P}_*(A)$$

makes

$$(A,\ \vee,\ \wedge,\ 0,\ 1,\ ')$$

a Boolean hyperalgebra in the sense of Definition (Boolean hyperalgebra) above.

Remark 3.20.5.

1. When $n = 1$, $\vee^{(1)} = \vee$ and a Boolean 1-SuperHyperalgebra is exactly a Boolean hyperalgebra.

2. The requirement via $\flat_n$ is a clean way to ensure that the nested outputs of $\vee^{(n)}$ are *Boolean* at the level of observable (flattened) joins.

Example 3.20.6 (Real-world computer-science example of a Boolean n-SuperHyperalgebra)**.** Consider a compiler toolchain that transforms access-control policies through multiple stages:

$$\text{(high-level rules)}\ \longrightarrow\ \text{(intermediate normal forms)}\ \longrightarrow\ \text{(low-level engine code)}.$$

At each stage, the compiler may generate *multiple* semantically equivalent artifacts (e.g. because of different optimization passes, rule reordering, or target-specific backends), and those alternatives may branch again at the next stage. This creates a *hierarchy of choices*.

Let A again be the set of deployable (final) policy artifacts, with $0, 1, '$ interpreted as in Example 3.20.3, and keep $\wedge$ as a deterministic conjunction at the final level. For a fixed $n \geq 2$, define a superhyperoperation

$$\vee^{(n)} : A \times A \to \mathcal{P}^n_*(A)$$

as follows: for two policies $a, b \in A$, the value $a \vee^{(n)} b$ is an n-nested nonempty collection whose elements encode *stage-wise families* of admissible merged outputs. Informally, the outer levels represent choices of compiler/back-end families, while inner levels represent choices of optimizations and rewrites within a family, ultimately yielding concrete deployable artifacts in A.

Flattening the nested set of alternatives,

$$a \vee b \;:=\; \flat_n\big(a \vee^{(n)} b\big) \in \mathcal{P}_*(A),$$

produces exactly the set of all deployable merged policies reachable through *some* chain of compilation choices. Requiring that $(A, \vee, \wedge, 0, 1, ')$ is a Boolean hyperalgebra then guarantees that, despite the multi-level branching, the induced merge behavior at the deployable level satisfies the same Boolean-style correctness constraints as in Example 3.20.3. Hence $(A, \vee^{(n)}, \wedge, 0, 1, ')$ models a Boolean n-SuperHyperalgebra capturing *multi-stage, hierarchical nondeterminism* in policy compilation and deployment.

Chapter 4

HyperStructure and SuperHyperStructure in Discrete Mathematics and Combinatorics

In this chapter, we investigate hyperstructures and superhyperstructures in discrete mathematics and combinatorics. Discrete mathematics studies countable structures like graphs, logic, sets, and algorithms, focusing on finite or enumerable objects and proofs [192]. Combinatorics investigates counting, arrangements, and structures; it analyzes permutations, combinations, graphs, and extremal properties of finite systems [193]. For reference, Table 4.1 summarizes classical concepts in discrete mathematics and combinatorics together with their hyper and superhyper counterparts.

4.1 HyperGraph and SuperHyperGraph

Graphs and hypergraphs are fundamental combinatorial models for discrete structures. A classical (undirected) graph can be viewed as a special case of a hypergraph in which every edge contains exactly two vertices [194]. In contrast, a classical hypergraph permits an edge to connect an arbitrary (finite) number of vertices, making it suitable for representing multiway relationships [195–197]. As related concepts of hypergraphs, directed hypergraphs [198,199], fuzzy hypergraphs [200–202], soft hypergraphs [203, 204], and neutrosophic hypergraphs [205, 206] are also known.

Definition 4.1.1 (Hypergraph [196, 207])**.** A *hypergraph* is a pair $H = (V(H), E(H))$ where

$$V(H) \neq \varnothing \quad \text{and} \quad E(H) \subseteq \mathcal{P}^*\big(V(H)\big).$$

Throughout this book both $V(H)$ and $E(H)$ are assumed to be finite. Elements of $V(H)$ are called *vertices*, and elements of $E(H)$ are called *hyperedges*.

Example 4.1.2 (A small hypergraph)**.** Let $V(H) = \{1, 2, 3, 4\}$. Define the hyperedge family

$$E(H) = \{\, e_1, e_2, e_3 \,\} \quad \text{with} \quad e_1 = \{1, 2\},\ e_2 = \{2, 3, 4\},\ e_3 = \{1, 3\}.$$

Table 4.1: Classical concepts and their hyper/superhyper counterparts in discrete mathematics and combinatorics.

Classical Concept	**HyperConcept**	**SuperHyperConcept**
Graph	HyperGraph	SuperHyperGraph
Game	HyperGame	SuperHyperGame
Automaton	HyperAutomaton	SuperHyperAutomaton
Formal language	HyperLanguage	SuperHyperLanguage
Algorithm	HyperAlgorithm	SuperHyperAlgorithm
Floorplan (VLSI / layout)	HyperFloorplan	SuperHyperFloorplan

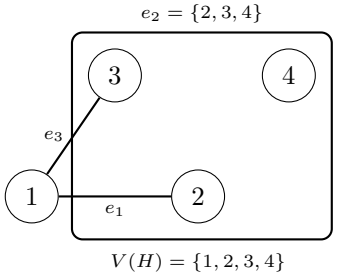


Figure 4.1: A small hypergraph H with hyperedges $e_1 = \{1, 2\}$, $e_2 = \{2, 3, 4\}$, and $e_3 = \{1, 3\}$.

Equivalently, one may write

$$H = (V(H), E(H)) \quad \text{with} \quad E(H) \subseteq \mathcal{P}^*(V(H)).$$

Here e_2 is a genuine hyperedge of size 3, while e_1, e_3 are ordinary edges of size 2. Thus H is a finite hypergraph in the sense of the definition. For reference, the graph of this concrete example is shown in Fig. 4.1.

A *SuperHyperGraph* carries this idea further by forming vertices and edges from iterated powersets of a base set; this viewpoint has appeared in several recent contexts [15, 17, 208, 209]. Reported applications include, among others, molecular structure modeling, complex network analysis, and signal processing [35, 210–213]. As an extension, concepts such as Plithogenic SuperHyperGraphs are also known [214–217].

Definition 4.1.3 (n-SuperHyperGraph)**.** [15,16,35] Fix a finite base set V_0 and a level $n \in \mathbb{N}_0$. An *n-SuperHyperGraph over V_0* is a triple

$$\mathrm{SHG}^{(n)} \;=\; \big(V,\; E,\; \partial\big),$$

where

- $V \subseteq \mathcal{P}^{\,n}(V_0)$ is a finite set of *n-supervertices*;
- $E \subseteq \mathcal{P}(V)$ is a finite set of *(super)edge identifiers*;
- $\partial : E \to \mathcal{P}^*(V)$ is an *incidence map* sending each edge to a nonempty finite subset of V.

For $e \in E$, the set $\partial(e) \subseteq V$ is called the *(super)edge incidence set.*

Example 4.1.4 (A 2-SuperHyperGraph over a three-element base set)**.** Let the base set be

$$V_0 = \{a, b, c\}.$$

Fix $n = 2$. Then $\mathcal{P}^2(V_0) = \mathcal{P}(\mathcal{P}(V_0))$, so a 2-supervertex is a set of subsets of V_0.

Choose three 2-supervertices:

$$v_1 := \big\{\{a\}, \{a, b\}\big\}, \qquad v_2 := \big\{\{b\}, \{b, c\}\big\}, \qquad v_3 := \big\{\{c\}\big\}.$$

Let

$$V = \{v_1, v_2, v_3\} \subseteq \mathcal{P}^2(V_0).$$

Define two superedge identifiers $E = \{e_1, e_2\} \subseteq \mathcal{P}(V)$ and an incidence map $\partial : E \to \mathcal{P}^*(V)$ by

$$\partial(e_1) = \{v_1, v_2\}, \qquad \partial(e_2) = \{v_1, v_2, v_3\}.$$

Then $\partial(e_i)$ is a nonempty finite subset of V for each i, so ∂ is a valid incidence map.

Hence

$$\mathrm{SHG}^{(2)} = \big(V, E, \partial\big)$$

is a concrete 2-SuperHyperGraph over V_0 in the sense of Definition 4.1.3. Intuitively, v_1, v_2, v_3 represent *clusters of clusters* of base vertices, and each superedge connects several such clusters at once via its incidence set.

A useful way to view an (m, n)-SuperHyperGraph is as a *hypergraph of transformers*. Vertices are not atomic points but concrete set-to-set operators $f : \mathcal{P}_m(S) \to \mathcal{P}_n(S)$ defined on a fixed base set S, while each hyperedge is a finite nonempty family of such operators, intended to encode a shared context, compatibility constraint, or joint interaction among the corresponding rules. The (h, k)-ary extension replaces unary transformers by multi-input/multi-output ones, namely maps $(\mathcal{P}_m(S))^h \to (\mathcal{P}_n(S))^k$, thereby allowing a vertex to combine several m-level sources and produce k output channels [218].

Notation 4.1.5. *For a nonempty base set S define iterated powersets by*

$$\mathcal{P}_0(S) := S, \qquad \mathcal{P}_{m+1}(S) := \mathcal{P}\big(\mathcal{P}_m(S)\big) \quad (m \in \mathbb{N}_0),$$

so that $\mathcal{P}_1(S) = \mathcal{P}(S)$, $\mathcal{P}_2(S) = \mathcal{P}(\mathcal{P}(S))$, etc. We also write $X^h := \underbrace{X \times \cdots \times X}_{h \text{ copies}}$ for $h \in \mathbb{N}$.

Definition 4.1.6 ((m, n)-*superhyperfunction*)**.** [219,220] Let $m, n \in \mathbb{N}$ and $S \neq \varnothing$. An (m, n)-*superhyperfunction on S* is a map

$$f : \ \mathcal{P}_m(S) \longrightarrow \mathcal{P}_n(S).$$

Definition 4.1.7 ((m, n)-*SuperHyperGraph*)**.** [218] Fix $m, n \in \mathbb{N}$ and a nonempty base set S. Let

$$\mathfrak{F}_{m,n}(S) \ := \ \big\{\, f : \mathcal{P}_m(S) \to \mathcal{P}_n(S) \,\big\}.$$

An (m,n)-*SuperHyperGraph* is a pair

$$\mathsf{SHG}^{(m,n)} = (V, \mathcal{E}),$$

where $V \subseteq \mathfrak{F}_{m,n}(S)$ is a nonempty set of vertices (each vertex is a concrete (m,n)-superhyperfunction) and

$$\varnothing \neq \mathcal{E} \ \subseteq \ \mathcal{P}(V) \setminus \{\varnothing\}$$

is a nonempty family of nonempty hyperedges. Each $E \in \mathcal{E}$ is interpreted as a finite collection of superhyperfunctions that are to be considered jointly (e.g., as mutually consistent rules or as a context-dependent bundle).

Example 4.1.8 (A $(1,2)$-SuperHyperGraph of transformer-vertices)**.** Let $S = \{a, b\}$ and take $(m,n) = (1,2)$. Then

$$\mathcal{P}_1(S) = \mathcal{P}(S), \qquad \mathcal{P}_2(S) = \mathcal{P}\big(\mathcal{P}(S)\big).$$

Hence a vertex is a concrete map $f : \mathcal{P}(S) \to \mathcal{P}(\mathcal{P}(S))$.

Vertices (two $(1,2)$-superhyperfunctions). Define $f, g : \mathcal{P}(S) \to \mathcal{P}_2(S)$ by

$$f(A) := \{A,\ S \setminus A\}, \qquad g(A) := \{A,\ A \cup \{a\}\}, \qquad (A \subseteq S).$$

Each value is a set whose elements are subsets of S; thus $f(A), g(A) \in \mathcal{P}(\mathcal{P}(S)) = \mathcal{P}_2(S)$. Therefore $f, g \in \mathfrak{F}_{1,2}(S)$.

Set

$$V := \{f, g\} \subseteq \mathfrak{F}_{1,2}(S).$$

Hyperedges (contexts bundling vertices). Define a nonempty family of nonempty hyperedges by

$$\mathcal{E} := \{E_1, E_2\}, \qquad E_1 := \{f\}, \qquad E_2 := \{f, g\}.$$

Then $\varnothing \neq \mathcal{E} \subseteq \mathcal{P}(V) \setminus \{\varnothing\}$.

Concrete evaluation. For $A = \{b\} \subseteq S$, one has

$$f(\{b\}) = \big\{\{b\}, \{a\}\big\}, \qquad g(\{b\}) = \big\{\{b\}, \{a, b\}\big\}.$$

Thus the hyperedge $E_2 = \{f, g\}$ represents a context in which the two transformers f and g are considered jointly.

Therefore $\mathsf{SHG}^{(1,2)} = (V, \mathcal{E})$ is a concrete (m,n)-SuperHyperGraph in the sense of Definition 4.1.7.

For convenience, a compact comparison between an n-SuperHyperGraph and an (m,n)-SuperHyperGraph is presented in Table 4.2.

Table 4.2: Compact comparison: n-SuperHyperGraph vs. (m,n)-SuperHyperGraph.

Item	n-SuperHyperGraph	(m,n)-SuperHyperGraph
Base data	A finite base set V_0 (universe for iterated powersets).	A nonempty base set S (universe for iterated powersets).
Vertex type	Supervertices are n-level objects: $V \subseteq \mathcal{P}^n(V_0)$.	Vertices are concrete (m,n)-superhyperfunctions: $V \subseteq \mathfrak{F}_{m,n}(S) = \{f : \mathcal{P}_m(S) \to \mathcal{P}_n(S)\}$.
Edge / incidence model	Identifiers E with an incidence map $\partial : E \to \mathcal{P}(V) \setminus \{\varnothing\}$.	Hyperedges are subsets of V: $\varnothing \neq \mathcal{E} \subseteq \mathcal{P}(V) \setminus \{\varnothing\}$. (Equivalently, encode each hyperedge by an identifier plus an incidence map.)
What is "higher level"?	Higher level is in the *objects* (vertices live in $\mathcal{P}^n(V_0)$).	Higher level is in the *morphisms* (vertices are maps $\mathcal{P}_m(S) \to \mathcal{P}_n(S)$).
Typical reading	Each supervertex represents an aggregated entity; each superedge links multiple supervertices by incidence.	Each vertex is a set-to-set transformer; each hyperedge groups transformers to encode joint constraints or contexts.
Relation to classical hypergraphs	Recovered by taking $n = 0$ and $V \subseteq V_0$.	Shifts the focus from n-level sets to set-valued transformers as vertices (Definition 4.1.7).

Definition 4.1.9 ((h,k)-*ary* (m,n)-*superhyperfunction*)**.** [220] Let $h,k \in \mathbb{N}$ and $m,n \in \mathbb{N}$. An (h,k)-*ary* (m,n)-*superhyperfunction on* S is a map

$$F : \ (\mathcal{P}_m(S))^h \longrightarrow (\mathcal{P}_n(S))^k.$$

Writing $F(A_1,\dots,A_h) = (B_1,\dots,B_k)$, each component $B_j \in \mathcal{P}_n(S)$ is an n-level object determined by h many m-level inputs.

Definition 4.1.10 ((h,k)-*ary* (m,n)-*SuperHyperGraph*)**.** [218] Fix $m,n,h,k \in \mathbb{N}$ and $S \neq \varnothing$. Let

$$\mathfrak{F}^{h,k}_{m,n}(S) := \left\{ F : (\mathcal{P}_m(S))^h \to (\mathcal{P}_n(S))^k \right\}.$$

An (h,k)-*ary* (m,n)-*SuperHyperGraph* is a pair

$$\mathsf{SHG}^{(h,k)}_{(m,n)} = (V,\mathcal{E}), \qquad \varnothing \neq V \subseteq \mathfrak{F}^{h,k}_{m,n}(S), \quad \varnothing \neq \mathcal{E} \subseteq \mathcal{P}(V) \setminus \{\varnothing\}.$$

Example 4.1.11 (A $(2,2)$-ary $(1,1)$-SuperHyperGraph)**.** Let the base set be

$$S = \{a,b,c\}, \qquad m = n = 1, \qquad h = k = 2.$$

Then $\mathcal{P}_1(S) = \mathcal{P}(S)$, so a vertex is a concrete map

$$F : \big(\mathcal{P}(S)\big)^2 \longrightarrow \big(\mathcal{P}(S)\big)^2.$$

Two vertices (two transformer-vertices). Define two $(2,2)$-ary $(1,1)$-superhyperfunctions F_1, F_2 by

$$F_1(A,B) := \big(A \cup B,\ A \cap B\big), \qquad F_2(A,B) := \big(A \triangle B,\ B \setminus A\big),$$

Table 4.3: Compact comparison: (m,n)-SuperHyperGraph vs. (h,k)-ary (m,n)-SuperHyperGraph.

Category	(m,n)-SuperHyperGraph $\mathsf{SHG}^{(m,n)} = (V,\mathcal{E})$	(h,k)-ary (m,n)-SuperHyperGraph $\mathsf{SHG}^{(h,k)}_{(m,n)} = (V,\mathcal{E})$
Underlying base data	A nonempty base set S and levels $m,n \in \mathbb{N}$.	A nonempty base set S, levels $m,n \in \mathbb{N}$, and arities $h,k \in \mathbb{N}$.
Vertex meaning	Each vertex is a concrete (m,n)-superhyperfunction $f : \mathcal{P}_m(S) \to \mathcal{P}_n(S)$.	Each vertex is a concrete (h,k)-ary (m,n)-superhyperfunction $F : (\mathcal{P}_m(S))^h \to (\mathcal{P}_n(S))^k$.
Input–output interface	One m-level input $A \in \mathcal{P}_m(S)$ mapped to one n-level output $f(A) \in \mathcal{P}_n(S)$.	h inputs $(A_1,\ldots,A_h)$ mapped to k outputs $F(A_1,\ldots,A_h) = (B_1,\ldots,B_k)$.
Interpretation	Unary rule/transformer: one source $\mapsto$ one hierarchical output.	Multi-input, multi-output transformer: fuses h sources and returns k output channels.
Hyperedges	$\varnothing \neq \mathcal{E} \subseteq \mathcal{P}(V) \setminus \{\varnothing\}$; each hyperedge groups finitely many vertices (rules).	Same formalism for $\mathcal{E}$, but grouped vertices are multi-ary, multi-output transformers.
Specialization	Recovered as the case $(h,k) = (1,1)$.	Strictly generalizes the unary model when $h \neq 1$ and/or $k \neq 1$.

for all $A, B \subseteq S$, where $A \triangle B = (A \setminus B) \cup (B \setminus A)$ is the symmetric difference. Clearly, each output component is a subset of S, hence $F_1, F_2 \in \mathfrak{F}^{2,2}_{1,1}(S)$.

Set the vertex set

$$V := \{F_1, F_2\} \;\subseteq\; \mathfrak{F}^{2,2}_{1,1}(S).$$

Hyperedges (contexts bundling vertices). Define a nonempty family of nonempty hyperedges by

$$\mathcal{E} := \{\, E_1, E_2 \,\}, \qquad E_1 := \{F_1\}, \qquad E_2 := \{F_1, F_2\}.$$

Then $\varnothing \neq \mathcal{E} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$.

Concrete evaluation (interpretation). For example, with $A = \{a,b\}$ and $B = \{b,c\}$,

$$F_1(A,B) = \big(\{a,b,c\}, \{b\}\big), \qquad F_2(A,B) = \big(\{a,c\}, \{c\}\big).$$

Thus F_1 outputs two channels (union and intersection), while F_2 outputs two other channels (symmetric difference and the "B minus A" part). The hyperedge $E_2 = \{F_1, F_2\}$ represents a context in which these two transformers are considered jointly.

Therefore

$$\mathsf{SHG}^{(2,2)}_{(1,1)} = (V,\mathcal{E})$$

is a concrete (h,k)-ary (m,n)-SuperHyperGraph in the sense of Definition 4.1.10.

For convenience, Table 4.3 summarizes the difference between the unary and the arity-extended models.

4.2 Hierarchical SuperHyperGraph

A hierarchical superhypergraph is a superhypergraph whose vertices live across multiple powerset levels, with edges allowed to join mixed-level supervertices, while maintaining downward-closure coherence (cf. [221, 222]).

Definition 4.2.1 (Support map and edge-support)**.** Define the support map $\mathrm{Supp} : V \to \mathcal{P}(V_0) \setminus \{\emptyset\}$ recursively by

$$\mathrm{Supp}(v) := \{v\} \quad (v \in V_0), \qquad \mathrm{Supp}(X) := \bigcup_{y \in X} \mathrm{Supp}(y) \quad (X \in V \setminus V_0),$$

and for each $e \in E$ define its *base support* (a hyperedge on V_0) by

$$\sigma(e) \ := \bigcup_{x \in e} \mathrm{Supp}(x) \ \in \ \mathcal{P}(V_0) \setminus \{\emptyset\}.$$

Definition 4.2.2 (Hierarchical SuperHyperGraph of height r)**.** Let V_0 be a finite, nonempty base set. For $k \geq 0$ define iterated powersets

$$\mathcal{P}^0(V_0) := V_0, \qquad \mathcal{P}^{k+1}(V_0) := \mathcal{P}\big(\mathcal{P}^k(V_0)\big),$$

and fix an integer $r \geq 0$. Set the *hierarchical universe*

$$\mathcal{U}_r(V_0) \ := \bigcup_{k=0}^{r} \Big(\mathcal{P}^k(V_0) \setminus \{\emptyset\}\Big).$$

For $x \in \mathcal{U}_r(V_0)$, define its *level*

$$\ell(x) \ := \ \min\{\, k \in \{0, 1, \ldots, r\} : x \in \mathcal{P}^k(V_0) \,\}.$$

A *hierarchical superhypergraph of height* r on V_0 is a pair

$$\mathbb{H}^{\langle r \rangle} = (V, E)$$

such that

(H1) (*Hierarchical vertex set*) V is a finite nonempty set with

$$V \subseteq \mathcal{U}_r(V_0).$$

Elements of V are called *hierarchical supervertices*.

(H2) (*Cross-level edges*) E is a finite family of nonempty subsets of V:

$$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

Elements of E are called *hierarchical superhyperedges*. In particular, a superhyperedge may contain supervertices of *different* levels.

(H3) (*Coherence / downward closure*) If $X \in V$ and $\ell(X) \geq 1$, then

$$X \subseteq V.$$

Equivalently, whenever a higher-level supervertex is present, all its immediate constituents are also present as supervertices.

For each $k \in \{0, \dots, r\}$ we define the *k-th layer* by

$$V_k \;:=\; \{\, x \in V :\; \ell(x) = k \,\}, \qquad \text{so that} \qquad V = \dot{\bigcup}_{k=0}^{r} V_k.$$

Example 4.2.3 (A hierarchical SuperHyperGraph of height $r = 2$)**.** Let the base set be

$$V_0 = \{a, b, c\}, \qquad r = 2.$$

Then

$$\mathcal{U}_2(V_0) = \Big(V_0\Big) \;\cup\; \Big(\mathcal{P}(V_0) \setminus \{\emptyset\}\Big) \;\cup\; \Big(\mathcal{P}^2(V_0) \setminus \{\emptyset\}\Big).$$

Hierarchical vertex set. Define the following supervertices:

$$V_0^* = \{a, b, c\},$$
$$x_{ab} := \{a, b\}, \quad x_{bc} := \{b, c\}, \quad x_{abc} := \{a, b, c\},$$
$$X_1 := \{\{a\}, \{a, b\}\} = \big\{\{a\}, x_{ab}\big\}, \qquad X_2 := \{\{b, c\}, \{a, b, c\}\} = \big\{x_{bc}, x_{abc}\big\}.$$

Set

$$V := \{a, b, c,\; x_{ab}, x_{bc}, x_{abc},\; X_1, X_2\}.$$

Then $V \subseteq \mathcal{U}_2(V_0)$ and $V \neq \emptyset$.

Levels and layers. By construction,

$$\ell(a) = \ell(b) = \ell(c) = 0, \qquad \ell(x_{ab}) = \ell(x_{bc}) = \ell(x_{abc}) = 1, \qquad \ell(X_1) = \ell(X_2) = 2.$$

Hence

$$V_0 = \{a, b, c\}, \quad V_1 = \{x_{ab}, x_{bc}, x_{abc}\}, \quad V_2 = \{X_1, X_2\}.$$

Coherence (downward closure). Since $X_1 = \{\{a\}, x_{ab}\}$ and $X_2 = \{x_{bc}, x_{abc}\}$, we have

$$X_1 \subseteq V, \qquad X_2 \subseteq V,$$

so condition (H3) holds: every immediate constituent of each level-2 supervertex is included in V.

Cross-level edges. Define a finite family of hierarchical superhyperedges $E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$ by

$$e_1 := \{\, a,\; x_{ab},\; X_1 \,\}, \qquad e_2 := \{\, b,\; x_{bc},\; X_2 \,\}, \qquad e_3 := \{\, X_1,\; X_2,\; x_{abc} \,\}.$$

Set $E := \{e_1, e_2, e_3\}$. Each e_i is nonempty and may mix levels $0, 1, 2$, so (H2) is satisfied.

Support map and edge-support. The support map $\mathrm{Supp} : V \to \mathcal{P}(V_0) \setminus \{\emptyset\}$ is computed recursively:

$$\mathrm{Supp}(a) = \{a\},\; \mathrm{Supp}(b) = \{b\},\; \mathrm{Supp}(c) = \{c\},$$
$$\mathrm{Supp}(x_{ab}) = \mathrm{Supp}(\{a, b\}) = \{a, b\}, \quad \mathrm{Supp}(x_{bc}) = \{b, c\}, \quad \mathrm{Supp}(x_{abc}) = \{a, b, c\},$$
$$\mathrm{Supp}(X_1) = \mathrm{Supp}(\{a\}) \cup \mathrm{Supp}(x_{ab}) = \{a\} \cup \{a, b\} = \{a, b\},$$
$$\mathrm{Supp}(X_2) = \mathrm{Supp}(x_{bc}) \cup \mathrm{Supp}(x_{abc}) = \{b, c\} \cup \{a, b, c\} = \{a, b, c\}.$$

Hence the base supports $\sigma(e) = \bigcup_{x \in e} \mathrm{Supp}(x)$ are:

$$\sigma(e_1) = \mathrm{Supp}(a) \cup \mathrm{Supp}(x_{ab}) \cup \mathrm{Supp}(X_1) = \{a\} \cup \{a, b\} \cup \{a, b\} = \{a, b\},$$
$$\sigma(e_2) = \mathrm{Supp}(b) \cup \mathrm{Supp}(x_{bc}) \cup \mathrm{Supp}(X_2) = \{b\} \cup \{b, c\} \cup \{a, b, c\} = \{a, b, c\},$$
$$\sigma(e_3) = \mathrm{Supp}(X_1) \cup \mathrm{Supp}(X_2) \cup \mathrm{Supp}(x_{abc}) = \{a, b\} \cup \{a, b, c\} \cup \{a, b, c\} = \{a, b, c\}.$$

Therefore $\mathbb{H}^{\langle 2 \rangle} = (V, E)$ is a hierarchical SuperHyperGraph of height $r = 2$ in the sense of Definition 4.2.2, with the support and edge-support given by Definition 4.2.1.

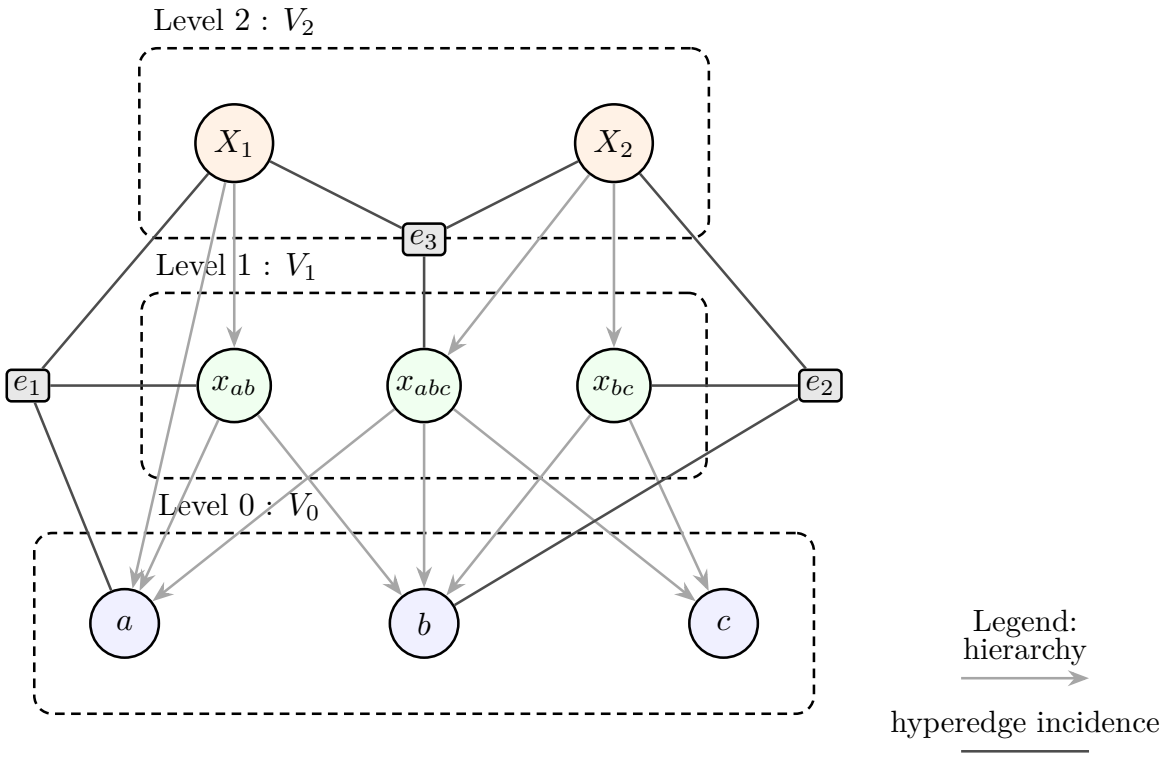


Figure 4.2: A hierarchical SuperHyperGraph of height $r = 2$ for $V = \{a, b, c, x_{ab}, x_{bc}, x_{abc}, X_1, X_2\}$ and $E = \{e_1, e_2, e_3\}$, where $e_1 = \{a, x_{ab}, X_1\}$, $e_2 = \{b, x_{bc}, X_2\}$, $e_3 = \{X_1, X_2, x_{abc}\}$.

4.3 Recursive SuperHyperGraph

A *Recursive HyperGraph* is a hypergraph whose hyperedges may contain ordinary vertices and also lower-level hyperedges as elements, enabling nested, self-referential incidence up to a bounded recursion depth [223, 224].

Definition 4.3.1 (Depth-k powerset universe)**.** [223, 224] Let S be a (nonempty) set and let $k \in \mathbb{N} \cup \{0\}$. Define a hierarchy of sets $(S_i)_{i \geq 0}$ by

$$S_0 := S, \qquad S_i := \mathcal{P}\Big(\bigcup_{j=0}^{i-1} S_j\Big) \quad (i \geq 1).$$

The *depth-k powerset universe* over S is

$$2_{S,k} := \mathcal{P}\Big(\bigcup_{i=0}^{k} S_i\Big).$$

Definition 4.3.2 (k-recursive hypergraph)**.** [223, 224] Let V be a (finite) vertex set and let $k \in \mathbb{N} \cup \{0\}$. A *$k$-recursive hypergraph* is a pair

$$H = (V, E)$$

such that

$$E \subseteq 2_{V,k} \setminus \{\emptyset\},$$

where $2_{V,k}$ is the depth-k powerset universe from Definition 4.3.1 applied to $S = V$.

In particular, for $k = 0$ one has $2_{V,0} = \mathcal{P}(V)$ and thus $E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$, i.e., H reduces to an ordinary hypergraph.

Example 4.3.3 (A 1-recursive hypergraph)**.** Let $V = \{a, b, c\}$ and take $k = 1$. By Definition 4.3.1 (with $S = V$),

$$S_0 = V, \qquad S_1 = \mathcal{P}(S_0) = \mathcal{P}(V), \qquad 2_{V,1} = \mathcal{P}(S_0 \cup S_1) = \mathcal{P}\big(V \cup \mathcal{P}(V)\big).$$

Hence an admissible edge at depth 1 may contain ordinary vertices (elements of V) and also *vertex-sets* (elements of $\mathcal{P}(V)$).

Define three nonempty edges:

$$e_1 := \{a, b\} \in \mathcal{P}(V) \subseteq V \cup \mathcal{P}(V),$$

$$e_2 := \{\, c,\ \{a, b\}\,\} \subseteq V \cup \mathcal{P}(V),$$

$$e_3 := \{\, \{b\},\ \{a, c\}\,\} \subseteq V \cup \mathcal{P}(V).$$

Here e_2 is genuinely recursive, since it contains the vertex c and also the set $\{a, b\} \in \mathcal{P}(V)$.

Let

$$E := \{e_1, e_2, e_3\}.$$

Then each $e_i \neq \emptyset$ and $E \subseteq 2_{V,1} \setminus \{\emptyset\}$. Therefore

$$H = (V, E)$$

is a concrete 1-recursive hypergraph in the sense of Definition 4.3.2.

An (n, k)-recursive SuperHyperGraph has level-n supervertices (iterated powersets) and depth-k recursive edges that may include supervertices and lower-level edges as elements. We restrict to well-founded recursive superhyperedges (no membership cycles).

Definition 4.3.4 ((n, k)-recursive SuperHyperGraph)**.** Fix a base (ground) set V_0 and let $n, k \in \mathbb{N} \cup \{0\}$.

(Iterated powersets). Define the iterated powersets by

$$\mathcal{P}^0(V_0) = V_0, \qquad \mathcal{P}^{n+1}(V_0) = \mathcal{P}\big(\mathcal{P}^n(V_0)\big) \quad (n \geq 0).$$

A (n, k)-*recursive SuperHyperGraph* is a pair

$$\mathrm{RSHG}^{(n,k)} = (V, E)$$

satisfying:

(i) *(Hierarchical supervertex set).* $V \subseteq \mathcal{P}^n(V_0)$.

(ii) *(Recursive superhyperedge family).* $E \subseteq 2_{V,k} \setminus \{\emptyset\}$, where $2_{V,k}$ is the depth-k powerset universe constructed from $S = V$ as in Definition 4.3.1.

Consistency checks. If $k = 0$, then $E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$ and one recovers the usual (non-recursive) n-SuperHyperGraph. If $n = 0$, then $V \subseteq V_0$ and one recovers a k-recursive hypergraph on the (ground) vertex set.

Example 4.3.5 (A $(2,1)$-recursive SuperHyperGraph)**.** Let the ground set be $V_0 = \{a, b\}$, and take $n = 2$ and $k = 1$. Then $\mathcal{P}^2(V_0) = \mathcal{P}(\mathcal{P}(V_0))$ consists of sets of subsets of V_0.

Supervertices. Choose two level-2 supervertices

$$v_1 := \{\{a\}\}, \qquad v_2 := \{\{a\}, \{b\}\}.$$

Set $V := \{v_1, v_2\} \subseteq \mathcal{P}^2(V_0)$.

Recursive edge universe at depth $k = 1$. From Definition 4.3.1 applied to $S = V$, we have

$$S_0 = V, \qquad S_1 = \mathcal{P}(S_0) = \mathcal{P}(V), \qquad 2_{V,1} = \mathcal{P}(S_0 \cup S_1) = \mathcal{P}\big(V \cup \mathcal{P}(V)\big).$$

Thus an admissible edge at depth 1 may contain *vertices* (elements of V) and also *vertex-sets* (elements of $\mathcal{P}(V)$).

Recursive superhyperedges. Define two edges:

$$e_1 := \{v_1, v_2\} \in \mathcal{P}(V) \subseteq V \cup \mathcal{P}(V),$$

$$e_2 := \{\, v_1,\ \{v_2\}\,\} \subseteq V \cup \mathcal{P}(V).$$

Note that e_2 is genuinely recursive: it contains the vertex v_1 and also the *subedge-like* object $\{v_2\} \in \mathcal{P}(V)$. Set

$$E := \{e_1, e_2\}.$$

Then each e_i is nonempty and $E \subseteq 2_{V,1} \setminus \{\emptyset\}$ holds.

Therefore $\mathrm{RSHG}^{(2,1)} = (V, E)$ is a concrete $(n, k) = (2, 1)$-recursive SuperHyperGraph in the sense of Definition 4.3.4.

Definition 4.3.6 ((m, n, k)-recursive SuperHyperGraph)**.** Fix a nonempty base set S and integers $m, n \in \mathbb{N}_0$ and $k \in \mathbb{N}_0$. Let $\mathcal{P}_t(S)$ be the iterated powersets from Notation 4.1.5, and let

$$\mathfrak{F}_{m,n}(S) \ := \ \{\, f : \mathcal{P}_m(S) \to \mathcal{P}_n(S)\,\}$$

be the class of (m, n)-superhyperfunctions on S.

A (m, n, k)-*recursive SuperHyperGraph on* S is a pair

$$\mathrm{RSHG}^{(m,n,k)}(S) \ = \ (V, E)$$

satisfying:

(i) **(Vertex set of transformers)** $V \subseteq \mathfrak{F}_{m,n}(S)$ is a finite nonempty set.

(ii) **(Depth-k recursive edge family)** E is a finite family of nonempty sets with

$$E \subseteq 2_{V,k} \setminus \{\emptyset\},$$

where $2_{V,k}$ is the depth-k powerset universe over V from Definition 4.3.1.

(iii) **(Well-foundedness)** Each $e \in E$ is well-founded under membership: there is no membership cycle $x_0 \in x_1 \in \cdots \in x_t = x_0$ inside the transitive closure of e.

Elements of V are called *functional supervertices* (or *transformer-vertices*), and elements of E are called *recursive superhyperedges*.

Remark 4.3.7. When $k = 0$, one has $2_{V,0} = \mathcal{P}(V)$, so edges are ordinary hyperedges $E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$. For $k \geq 1$, an edge may contain vertices and also lower-level "edge-like" objects (sets built from V), capturing nested incidence up to recursion depth k.

Example 4.3.8 (A $(1,1,1)$-recursive SuperHyperGraph of transformers)**.** Let the base set be $S = \{a, b\}$, and take $m = n = k = 1$. Then $\mathcal{P}_1(S) = \mathcal{P}(S) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$.

Vertices (transformers). Define two $(1,1)$-superhyperfunctions $f, g : \mathcal{P}(S) \to \mathcal{P}(S)$ by

$$f(A) := A \cup \{a\}, \qquad g(A) := A \cup \{b\} \qquad (A \subseteq S).$$

Let

$$V := \{f, g\} \subseteq \mathfrak{F}_{1,1}(S).$$

Recursive edge universe at depth $k = 1$. As before,

$$S_0 = V, \qquad S_1 = \mathcal{P}(V), \qquad 2_{V,1} = \mathcal{P}\big(V \cup \mathcal{P}(V)\big).$$

Recursive superhyperedges. Define two edges

$$e_1 := \{f, g\} \in \mathcal{P}(V) \subseteq V \cup \mathcal{P}(V), \qquad e_2 := \{\, f,\ \{g\}\,\} \subseteq V \cup \mathcal{P}(V).$$

Set $E := \{e_1, e_2\}$.

Well-foundedness. There is no membership cycle: e_2 contains $\{g\}$, but $\{g\}$ contains only g, and none of these sets contains e_2 again. Hence each edge is well-founded under membership.

Therefore $\mathrm{RSHG}^{(1,1,1)}(S) = (V, E)$ is a concrete $(m, n, k) = (1, 1, 1)$-recursive SuperHyperGraph in the sense of Definition 4.3.6.

Lemma 4.3.9 (Constant-vertex embedding)**.** *Let $V_0 \neq \emptyset$ be a base set and fix $n \in \mathbb{N}_0$. For each $X \in \mathcal{P}_n(V_0) \setminus \{\emptyset\}$, define the constant map*

$$c_X : V_0 = \mathcal{P}_0(V_0) \longrightarrow \mathcal{P}_n(V_0), \qquad c_X(v) := X \quad (v \in V_0).$$

Then $c_X \in \mathfrak{F}_{0,n}(V_0)$. Moreover, the assignment

$$\iota : \mathcal{P}_n(V_0) \setminus \{\emptyset\} \longrightarrow \mathfrak{F}_{0,n}(V_0), \qquad \iota(X) := c_X$$

is injective.

Proof. By definition, $\mathcal{P}_0(V_0) = V_0$, and c_X maps V_0 into $\mathcal{P}_n(V_0)$, so $c_X \in \mathfrak{F}_{0,n}(V_0)$. If $c_X = c_Y$, then evaluating at any $v \in V_0$ yields $X = c_X(v) = c_Y(v) = Y$, so ι is injective. □

Theorem 4.3.10 (Generalization to (m,n)-SuperHyperGraphs and (n,k)-recursive SuperHyperGraphs)**.** *Let $S \neq \emptyset$ and $m, n, k \in \mathbb{N}_0$.*

(a) ***(Recovering (m,n)-SuperHyperGraphs).*** *Every $(m,n,0)$-recursive SuperHyperGraph on S is exactly an (m,n)-SuperHyperGraph (in the sense of Definition 4.1.7).*

(b) ***(Embedding (n,k)-recursive SuperHyperGraphs).*** *Fix a finite base set V_0 and integers $n, k \in \mathbb{N}_0$. Let $\mathrm{RSHG}^{(n,k)} = (V, E)$ be an (n,k)-recursive SuperHyperGraph as in Definition 4.3.4, so $V \subseteq \mathcal{P}_n(V_0)$ and $E \subseteq 2_{V,k} \setminus \{\emptyset\}$. Assume $V \subseteq \mathcal{P}_n(V_0) \setminus \{\emptyset\}$ (i.e. no empty supervertex).*

Define

$$V^\sharp := \{\iota(X) \mid X \in V\} \subseteq \mathfrak{F}_{0,n}(V_0),$$

where $\iota(X) = c_X$ is the constant-vertex embedding from Lemma 4.3.9. Extend ι to a map $\widehat{\iota} : 2_{V,k} \to 2_{V^\sharp,k}$ recursively by:

$$\widehat{\iota}(x) := \iota(x) \quad (x \in V), \qquad \widehat{\iota}(A) := \{\widehat{\iota}(a) \mid a \in A\} \quad (A \text{ a set built from } V \text{ at any depth } \leq k).$$

Set

$$E^\sharp := \{\widehat{\iota}(e) \mid e \in E\}.$$

Then $\mathrm{RSHG}^{(0,n,k)}(V_0) := (V^\sharp, E^\sharp)$ is a $(0,n,k)$-recursive SuperHyperGraph in the sense of Definition 4.3.6, and $\widehat{\iota}$ induces an isomorphism between the recursive edge systems (V, E) and $(V^\sharp, E^\sharp)$. In particular, (m,n,k)-recursive SuperHyperGraphs generalize (n,k)-recursive SuperHyperGraphs up to canonical relabeling of vertices.

Proof. **(a)** If $k = 0$, then by Definition 4.3.1 we have

$$2_{V,0} = \mathcal{P}(V).$$

Hence the edge condition $E \subseteq 2_{V,0} \setminus \{\emptyset\}$ becomes $E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$, which is exactly the hyperedge condition in Definition 4.1.7. The vertex condition $V \subseteq \mathfrak{F}_{m,n}(S)$ is also identical. Therefore a $(m,n,0)$-recursive SuperHyperGraph is precisely an (m,n)-SuperHyperGraph.

(b) By Lemma 4.3.9, each $\iota(X) = c_X$ lies in $\mathfrak{F}_{0,n}(V_0)$, so $V^\sharp \subseteq \mathfrak{F}_{0,n}(V_0)$. Since V is finite and nonempty, so is $V^\sharp$.

Next, because $\iota : V \to V^\sharp$ is bijective onto its image and $\widehat{\iota}$ is defined by applying ι at depth 0 and then taking images level-by-level, $\widehat{\iota}$ is a bijection

$$\widehat{\iota}:\ 2_{V,k} \xrightarrow{\cong} 2_{V^\sharp,k}$$

(the inverse is obtained by replacing c_X back by X and again taking images recursively). Therefore $E^\sharp \subseteq 2_{V^\sharp,k} \setminus \{\emptyset\}$ is finite and consists of nonempty sets.

Well-foundedness is preserved because $\widehat{\iota}$ is an isomorphism of membership structures: a membership cycle among elements in the transitive closure of e exists if and only if the corresponding cycle exists in the transitive closure of $\widehat{\iota}(e)$. Hence $(V^\sharp, E^\sharp)$ satisfies all items of Definition 4.3.6 with $m = 0$, the same n, and the same recursion depth k. This shows that every (n, k)-recursive SuperHyperGraph (with nonempty supervertices) is canonically representable as a $(0, n, k)$-recursive SuperHyperGraph whose vertices are constant $(0, n)$-superhyperfunctions. □

4.4 HyperGame and SuperHyperGame

HyperGame is a collection of each player's subjective strategic games, capturing asymmetric beliefs about others' strategies and payoffs under adversarial uncertainty contexts [225–228].

Definition 4.4.1 (HyperGame)**.** A *HyperGame* is a collection

$$H \ = \ \{\, G_1,\, G_2,\, \ldots,\, G_n\},$$

where each player's perceived game is

$$G_i \ = \ (N,\ S_i,\ U_i),$$

with

- $N = \{1, 2, \ldots, n\}$ the set of players,
- $S_i = \{S_{i1}, S_{i2}, \ldots, S_{in}\}$ the strategy spaces as perceived by player i,
- $U_i = \{U_{i1}, U_{i2}, \ldots, U_{in}\}$ the utility functions as perceived by player i.

If for all $i \in N$ one has $S_i = S$ and $U_i = U$, then $H = \{G\}$ reduces to the classical game $G = (N, S, U)$.

Example 4.4.2 (A two-player HyperGame with asymmetric perceptions)**.** Let $N = \{1, 2\}$. The *objective* interaction is a Prisoner's-Dilemma-type situation, but each player misperceives the other's strategy set and payoff magnitudes.

Player 1's perceived game. Let player 1 believe that both players have only two strategies

$$S_{11} = S_{12} = \{C, D\},$$

and that payoffs are given by the usual Prisoner's Dilemma utilities

$$U_{11}(C,C) = 3,\ U_{11}(D,C) = 5,\ U_{11}(C,D) = 0,\ U_{11}(D,D) = 1,$$
$$U_{12}(C,C) = 3,\ U_{12}(D,C) = 0,\ U_{12}(C,D) = 5,\ U_{12}(D,D) = 1.$$

Thus player 1's perceived game is $G_1 = (N, S_1, U_1)$ with $S_1 = \{S_{11}, S_{12}\}$ and $U_1 = \{U_{11}, U_{12}\}$.

Player 2's perceived game. Let player 2 believe that player 1 has an additional "mixed" option M (e.g. a probabilistic or partially cooperative behavior), so that

$$S_{21} = \{C, D, M\}, \qquad S_{22} = \{C, D\}.$$

Player 2 also believes the payoffs are *less extreme* (e.g. due to risk aversion), such as

$$U_{21}(D,C) = 4,\ U_{21}(C,C) = 3,\ U_{21}(D,D) = 2,\ U_{21}(C,D) = 1,$$

and with M interpreted as an intermediate behavior:

$$U_{21}(M,C) = 3.5, \qquad U_{21}(M,D) = 2.0,$$

with U_{22} defined analogously for player 2's own payoffs. Hence player 2's perceived game is $G_2 = (N, S_2, U_2)$.

HyperGame. The corresponding HyperGame is the collection

$$H = \{G_1, G_2\},$$

where each G_i is the game as perceived by player i. Since $S_1 \neq S_2$ (and similarly $U_1 \neq U_2$), this does not collapse to a single classical game.

SuperHyperGame is a recursive, iterative hierarchy of games about games, modeling multiple levels of players' beliefs regarding others' perceived strategies and payoffs [229, 230].

Definition 4.4.3 (n-SuperHyperGame)**.** [230] Let $N = \{1, 2, \ldots, n\}$ be a finite set of players, and let

$$\Omega(N) = \big\{\, G = (N, S, U) \mid S = \{S_1, \ldots, S_n\},\ U = \{U_1, \ldots, U_n\}\big\}$$

denote the collection of all classical strategic games on N. Define the iterated powersets of $\Omega(N)$ by

$$P^0\big(\Omega(N)\big) = \Omega(N), \quad P^k\big(\Omega(N)\big) = \mathcal{P}\big(P^{k-1}(\Omega(N))\big) \quad (k \geq 1).$$

Then an *n-SuperHyperGame* is any element

$$H \ \in\ P^n\big(\Omega(N)\big),$$

that is, a set of $(n-1)$-SuperHyperGames. Equivalently, one may write

$$H = \{\, H^1,\, H^2,\, \ldots,\, H^n\}, \quad H^k \subseteq P^{k-1}\big(\Omega(N)\big),$$

where each H^k is the k-th level HyperGame and its members are perceived games

$$G_i^k = \big(N,\ S_i^k,\ U_i^k\big), \quad i \in N,$$

with S_i^k the strategy set and U_i^k the utility function at level k as perceived by player i.

In particular:

$$P^1\big(\Omega(N)\big) = \{\text{HyperGames on } N\},$$

and

$$P^2\big(\Omega(N)\big) = \{\text{sets of HyperGames}\},$$

and so on, up to the n-th level.

Example 4.4.4 (A 2-SuperHyperGame: beliefs about perceived games)**.** Let $N = \{1, 2\}$ and let $\Omega(N)$ be the set of classical strategic games on N. A 2-SuperHyperGame is an element of

$$P^2\big(\Omega(N)\big) = \mathcal{P}\big(\mathcal{P}(\Omega(N))\big),$$

i.e. a set of HyperGames.

Consider two possible "level-1" HyperGames (two different descriptions of the players' perceptions):

- $H^{(a)} = \{G_1^{(a)}, G_2^{(a)}\} \in P^1(\Omega(N))$, where $G_1^{(a)}, G_2^{(a)}$ are as in Example 4.4.2 (player 2 believes in the extra strategy M).

- $H^{(b)} = \{G_1^{(b)}, G_2^{(b)}\} \in P^1(\Omega(N))$, where both players now believe the other has only $\{C, D\}$, but player 1 believes payoffs are "high-stakes" while player 2 believes payoffs are "low-stakes" (i.e. $U_{11}^{(b)} \neq U_{21}^{(b)}$, etc.).

Define the 2-SuperHyperGame by

$$H = \{H^{(a)}, H^{(b)}\} \in P^2\big(\Omega(N)\big).$$

Interpretation: the external analyst (or a higher-order belief model) represents uncertainty between two distinct HyperGames, each of which already encodes first-order misperceptions by the players. Thus H is a concrete 2-SuperHyperGame: a set of HyperGames (games about perceived games).

4.5 HyperAutomata and SuperHyperAutomata

We introduce the concept of a *HyperAutomaton*, which is defined using the powerset construction. The formal definition is presented below (cf. [231]). Note that this definition has been refined based on the use of the powerset construction, improving upon the version originally given in [231].

Notation 4.5.1 (HyperAutomaton)**.** *Let V_0 be a finite set of* base states*. We define the* hyperstate space *as the powerset*

$$\mathcal{P}(V_0) = \{\, X \mid X \subseteq V_0\}.$$

Definition 4.5.2 (HyperAutomaton)**.** (cf. [231]) A *HyperAutomaton* is a quintuple

$$\mathcal{H} = \big(Q,\, \Sigma,\, \delta,\, q_0,\, F\big),$$

where

- $Q \subseteq \mathcal{P}(V_0)$ is the finite set of *hyperstates*.

- Σ is a finite *input alphabet.*
- $\delta : Q \times \Sigma \longrightarrow \mathcal{P}(Q)$ is the *hypertransition function*, which assigns to each hyperstate and input symbol a set of successor hyperstates.
- $q_0 \in Q$ is the *initial hyperstate.*
- $F \subseteq Q$ is the set of *accepting hyperstates.*

Example 4.5.3 (A HyperAutomaton recognizing "some member contains a")**.** Let the base set be $V_0 = \{a, b\}$, and let the input alphabet be $\Sigma = \{0, 1\}$. Define the hyperstate set $Q \subseteq \mathcal{P}(V_0)$ by

$$Q = \{\, q_\emptyset := \emptyset,\ q_a := \{a\},\ q_b := \{b\},\ q_{ab} := \{a, b\} \,\}.$$

Interpret a hyperstate $q \in Q$ as the *current set of possible active atoms.*

Choose the initial hyperstate $q_0 = q_\emptyset$, and let the accepting hyperstates be those containing a:

$$F = \{q_a, q_{ab}\}.$$

Define the hypertransition function $\delta : Q \times \Sigma \to \mathcal{P}(Q)$ by

$$\delta(q, 0) = \{q\} \qquad (q \in Q),$$

and

$$\delta(q, 1) = \begin{cases} \{q_a\}, & q = q_\emptyset, \\ \{q_{ab}\}, & q = q_b, \\ \{q_a\}, & q = q_a, \\ \{q_{ab}\}, & q = q_{ab}. \end{cases}$$

Thus input symbol 0 leaves the hyperstate unchanged, while input symbol 1 ensures that a is present (possibly together with b if b was already present).

Hence

$$\mathcal{H} = \big(Q, \Sigma, \delta, q_0, F\big)$$

is a concrete HyperAutomaton in the sense of the definition. For example, starting from $q_\emptyset$, the word 1 leads to $\delta(q_\emptyset, 1) = \{q_a\} \subseteq F$, so the word 1 is accepted, while the word 0 stays at $q_\emptyset \notin F$ and is rejected.

We introduce the concept of a *n-SuperHyperAutomaton*, which is defined using the powerset construction. The formal definition is presented below(cf. [231]). Note that this definition has been refined based on the use of the nth-powerset construction, improving upon the version originally given in [231].

Definition 4.5.4 (n-SuperHyperAutomaton)**.** (cf. [231]) An *n-SuperHyperAutomaton* is a quintuple

$$\mathcal{S}^{(n)} = \big(Q,\ \Sigma,\ \delta,\ q_0,\ F\big),$$

where

- $Q \subseteq \mathcal{P}^n(V_0)$ is the finite set of *n-superstates*.
- Σ is a finite *input alphabet*.
- $\delta : Q \times \Sigma \longrightarrow \mathcal{P}(Q)$ is the *superhypertransition function*, mapping each n-superstate and input symbol to a set of n-superstates.
- $q_0 \in Q$ is the *initial n-superstate*.
- $F \subseteq Q$ is the set of *accepting n-superstates*.

Example 4.5.5 (A 2-SuperHyperAutomaton whose superstates are sets of hyperstates)**.** Let $V_0 = \{a, b\}$ and $\Sigma = \{0, 1\}$ as in Example 4.5.3. Fix $n = 2$. Then

$$\mathcal{P}^2(V_0) = \mathcal{P}(\mathcal{P}(V_0))$$

so a 2-superstate is a set of subsets of V_0, i.e. a set of hyperstates.

Define a finite superstate set $Q \subseteq \mathcal{P}^2(V_0)$ by

$$Q = \Big\{ Q_0 := \{\emptyset\},\ Q_1 := \big\{\{a\}\big\},\ Q_2 := \big\{\{b\}\big\},\ Q_3 := \big\{\{a\}, \{b\}\big\},\ Q_4 := \big\{\{a, b\}\big\} \Big\}.$$

Intuitively, each Q_i is a *collection of possible hyperstates*.

Take the initial 2-superstate $q_0 = Q_0 = \{\emptyset\}$. Declare as accepting those superstates that contain at least one hyperstate containing a:

$$F = \{\, Q \in Q \mid \exists\, A \in Q \text{ with } a \in A \,\} = \{Q_1, Q_3, Q_4\}.$$

Define the superhypertransition function $\delta : Q \times \Sigma \to \mathcal{P}(Q)$ by

$$\delta(Q, 0) = \{Q\} \qquad (Q \in Q),$$

and, for input 1, by the "apply-a-insertion" rule:

$$\delta(Q, 1) = \Big\{\, \{A \cup \{a\} \mid A \in Q\} \,\Big\}.$$

This is well-defined because $\{A \cup \{a\} \mid A \in Q\} \in \mathcal{P}^2(V_0)$, and in fact lands in our chosen finite list:

$$\delta(Q_0, 1) = \Big\{\big\{\{a\}\big\}\Big\} = \{Q_1\}, \qquad \delta(Q_2, 1) = \Big\{\big\{\{a, b\}\big\}\Big\} = \{Q_4\}, \qquad \delta(Q_3, 1) = \Big\{\big\{\{a\}, \{a, b\}\big\}\Big\}.$$

(If desired, one may enlarge Q to include $\{\{a\}, \{a, b\}\}$ so that $\delta(Q_3, 1) \subseteq Q$ strictly.)

Hence

$$\mathcal{S}^{(2)} = \big(Q, \Sigma, \delta, q_0, F\big)$$

is a concrete 2-SuperHyperAutomaton in the sense of the definition: its states are elements of $\mathcal{P}^2(V_0)$, and transitions map to sets of such 2-superstates.

4.6 HyperLanguage and SuperHyperLanguage

A hyperword is a nonempty subset of all finite words over an alphabet, while a hyperlanguage is a collection of such nonempty subsets (hyperwords). The definitions and related concepts are outlined below [232].

Definition 4.6.1 (Hyperword and Hyperlanguage)**.** (cf. [233, 234]) Let Σ be a finite alphabet, and let Σ^* denote the set of all finite words over Σ.

1. A *hyperword* over Σ is a nonempty subset of Σ^*. In other words, a hyperword is an element of the power set $\mathcal{P}(\Sigma^*)$.

2. A *hyperlanguage* over Σ is a set of hyperwords over Σ. Thus, a hyperlanguage H is a subset of $\mathcal{P}(\Sigma^*)$. Formally:

$$H \subseteq \mathcal{P}(\Sigma^*).$$

A hyperlanguage can therefore be viewed as a *set of sets of words* over Σ.

Example 4.6.2 (Hyperword and hyperlanguage over a binary alphabet)**.** Let $\Sigma = \{0, 1\}$. Then Σ^* is the set of all finite binary strings.

Hyperword. Define the hyperword

$$W := \{\, 0,\ 1,\ 01,\ 10 \,\} \ \subseteq\ \Sigma^*.$$

This is a nonempty subset of Σ^*, hence a hyperword.

Hyperlanguage. Define a hyperlanguage $H \subseteq \mathcal{P}(\Sigma^*)$ by

$$H := \{\, W,\ W_{\text{even}},\ W_{\text{pal}} \,\},$$

where

$$W_{\text{even}} := \{\, w \in \Sigma^* \mid |w| \text{ is even} \,\}, \qquad W_{\text{pal}} := \{\, w \in \Sigma^* \mid w \text{ is a palindrome} \,\}.$$

Then H is a set of hyperwords (a set of sets of words), hence a hyperlanguage.

Definition 4.6.3 (Natural Hyperword and Natural Hyperlanguage)**.** [235, 236] Let $\mathcal{W}$ be the set of well-formed expressions (words, phrases, sentences) in a chosen natural language. Then:

1. A *natural hyperword* is any nonempty subset of $\mathcal{W}$; formally, it is an element of

$$\mathcal{P}(\mathcal{W}) \setminus \{\emptyset\}.$$

---

2. A *natural hyperlanguage* is a collection of natural hyperwords; that is, any set

$$\mathcal{HL}_{\text{nat}} \subseteq \mathcal{P}(\mathcal{W}).$$

Example 4.6.4 (Natural hyperword and natural hyperlanguage)**.** Let $\mathcal{W}$ be the set of well-formed English expressions (words/phrases/sentences). Consider the nonempty subset

$$W_{\text{nat}} := \{\text{“bank”, “river bank”, “financial bank”}\} \subseteq \mathcal{W}.$$

This is a *natural hyperword*, representing a small ambiguity cluster around the token “bank”.

Define a natural hyperlanguage by collecting several such ambiguity clusters:

$$\mathcal{HL}_{\text{nat}} := \left\{W_{\text{nat}},\ W_{\text{nat}}^{\text{charge}},\ W_{\text{nat}}^{\text{pitch}}\right\} \subseteq \mathcal{P}(\mathcal{W}),$$

where, for example,

$$W_{\text{nat}}^{\text{charge}} := \{\text{“charge”, “legal charge”, “battery charge”}\},$$

$$W_{\text{nat}}^{\text{pitch}} := \{\text{“pitch”, “musical pitch”, “sales pitch”}\}.$$

Thus $\mathcal{HL}_{\text{nat}}$ is a collection of natural hyperwords, i.e. a natural hyperlanguage.

An n-superhyperword is a nested set resulting from n iterated powersets of expressions; an n-superhyperlanguage is any collection of words. The definitions of n-Superhyperword and n-Superhyperlanguage are presented below [237–239].

Definition 4.6.5 (n-Superhyperword and n-Superhyperlanguage)**.** [239] We now generalize this construction to multiple levels. Define the iterated power sets as follows:

$$\mathcal{P}^0(\Sigma^*) := \Sigma^*, \quad \mathcal{P}^{k+1}(\Sigma^*) := \mathcal{P}(\mathcal{P}^k(\Sigma^*)), \text{ for all } k \geq 0.$$

1. An *n-superhyperword* over Σ is an element of $\mathcal{P}^n(\Sigma^*)$. In particular:

$$\mathcal{P}^1(\Sigma^*) = \mathcal{P}(\Sigma^*) \text{ consists of hyperwords,}$$

$$\mathcal{P}^2(\Sigma^*) = \mathcal{P}(\mathcal{P}(\Sigma^*)) \text{ consists of sets of hyperwords, and so forth.}$$

2. An *n-superhyperlanguage* over Σ is a subset of $\mathcal{P}^n(\Sigma^*)$. Formally:

$$L \subseteq \mathcal{P}^n(\Sigma^*).$$

Thus, an n-superhyperlanguage is a *set of* $(n-1)$*-superhyperwords*, generalizing the concept of a hyperlanguage to n-th level power sets of words.

Example 4.6.6 (A 2-superhyperword and a 2-superhyperlanguage)**.** Let $\Sigma = \{a, b\}$. A 2-superhyperword is an element of $\mathcal{P}^2(\Sigma^*) = \mathcal{P}(\mathcal{P}(\Sigma^*))$, i.e. a set of hyperwords.

Define two hyperwords:

$$W_1 := \{\, a,\ ab,\ aab \,\} \subseteq \Sigma^*, \qquad W_2 := \{\, b,\ ba,\ bba \,\} \subseteq \Sigma^*.$$

Then the set

$$\mathbb{W} := \{W_1, W_2\} \in \mathcal{P}\big(\mathcal{P}(\Sigma^*)\big) = \mathcal{P}^2(\Sigma^*)$$

is a concrete 2-superhyperword (a set of hyperwords).

Now define a 2-superhyperlanguage by taking a subset of $\mathcal{P}^2(\Sigma^*)$, for example

$$L^{(2)} := \{\ \mathbb{W},\ \mathbb{W}'\ \} \subseteq \mathcal{P}^2(\Sigma^*),$$

where $\mathbb{W}'$ is another 2-superhyperword, e.g.

$$\mathbb{W}' := \Big\{\ \{\, aa,\ aba \,\},\ \{\, bb,\ bab \,\}\ \Big\}.$$

Thus $L^{(2)}$ is a 2-superhyperlanguage: a set of 2-superhyperwords.

Definition 4.6.7 (Natural n-SuperHyperLanguage)**.** Let $\mathcal{W}$ be the set of well-formed expressions in a chosen natural language. We define iterated power sets:

$$\mathcal{P}^0(\mathcal{W}) := \mathcal{W}, \quad \mathcal{P}^{k+1}(\mathcal{W}) := \mathcal{P}\big(\mathcal{P}^k(\mathcal{W})\big),$$

for $k \geq 0$.

1. A *natural n-superhyperword* is any element of $\mathcal{P}^n(\mathcal{W})$. Concretely,

$$\mathcal{P}^1(\mathcal{W}) = \mathcal{P}(\mathcal{W}) \quad \text{(hyperwords)},$$
$$\mathcal{P}^2(\mathcal{W}) = \mathcal{P}\big(\mathcal{P}(\mathcal{W})\big), \quad \mathcal{P}^3(\mathcal{W}) = \mathcal{P}\Big(\mathcal{P}\big(\mathcal{P}(\mathcal{W})\big)\Big), \quad \ldots$$

2. A *Natural n-SuperHyperLanguage*, denoted $\mathcal{L}_n$, is defined as

$$\mathcal{L}_n \ \subseteq\ \mathcal{P}^n(\mathcal{W}).$$

   Hence, $\mathcal{L}_n$ is a collection of $(n-1)$-superhyperwords, but each level is taken over the set $\mathcal{W}$ of natural-language expressions rather than a purely formal alphabet.

Example 4.6.8 (A natural 2-superhyperword and natural 2-superhyperlanguage)**.** Let $\mathcal{W}$ be the set of well-formed English expressions. Define two natural hyperwords (level 1 objects):

$$W_{\text{bank}} := \{\text{“bank”},\ \text{“river bank”},\ \text{“financial bank”}\} \subseteq \mathcal{W},$$
$$W_{\text{bat}} := \{\text{“bat”},\ \text{“baseball bat”},\ \text{“bat (animal)”}\} \subseteq \mathcal{W}.$$

Then

$$\mathbb{W}_{\text{amb}} := \{W_{\text{bank}}, W_{\text{bat}}\} \in \mathcal{P}^2(\mathcal{W})$$

is a natural 2-superhyperword (a set of natural hyperwords).

Define a natural 2-superhyperlanguage by collecting such 2-superhyperwords:

$$\mathcal{L}_2 := \Big\{\mathbb{W}_{\text{amb}},\ \mathbb{W}_{\text{tech}}\Big\} \subseteq \mathcal{P}^2(\mathcal{W}),$$

where, for instance,

$$\mathbb{W}_{\text{tech}} := \Big\{\{\text{“port”},\ \text{“harbor port”},\ \text{“network port”}\},$$
$$\{\text{“cell”},\ \text{“biological cell”},\ \text{“battery cell”}\}\Big\} \in \mathcal{P}^2(\mathcal{W}).$$

Thus $\mathcal{L}_2$ is a natural 2-SuperHyperLanguage in the sense of Definition 4.6.7.

4.7 HyperAlgorithm and SuperHyperAlgorithm

A hyperalgorithm maps each input to a nonempty set of possible outputs or execution paths, representing nondeterministic computation explicitly [240]. A superhyperalgorithm iterates hyperalgorithmic outputs into nested powerset levels, modeling hierarchical nondeterminism and multi-stage uncertainty across computations [240]. The definition of a Hyperalgorithm is provided below.

Definition 4.7.1 (Hyperalgorithm)**.** [240] Let I and O be nonempty sets. Suppose that $\mathcal{H} = (\mathcal{P}(O), \circ)$ is a *hyperstructure* on O (i.e., the operation $\circ$ is defined on the powerset $\mathcal{P}(O)$). Let

$$\mathcal{A} = \{A_i : I \to O \mid i \in J\}$$

be a finite collection of algorithms, where each A_i is a total computable function. A *Hyperalgorithm* is a function

$$A : I \to \mathcal{P}(O)$$

defined by

$$A(x) = \circ\Big(\{A_i(x) : i \in J\}\Big) \quad \text{for all } x \in I.$$

In other words, a Hyperalgorithm aggregates the outputs of a family of algorithms via a hyperoperation $\circ$ on O. Notice that if $\circ$ is chosen to be degenerate (i.e., always returns a singleton), then the Hyperalgorithm reduces to a conventional multialgorithm.

Example 4.7.2 (A hyperalgorithm that aggregates two heuristics by a powerset-level hyperoperation)**.** Let $I = \mathbb{Z}$ and $O = \mathbb{Z}$. Consider two total (computable) algorithms $A_1, A_2 : I \to O$ defined by

$$A_1(x) := x, \qquad A_2(x) := -x.$$

Thus for any input x, the candidate outputs are $\{A_1(x), A_2(x)\} = \{x, -x\}$.

Define a powerset-level hyperoperation $\circ : \mathcal{P}(O) \to \mathcal{P}(O)$ by

$$\circ(S) := S \cup \{0\} \qquad (S \subseteq O).$$

(So $\circ$ is an operation on $\mathcal{P}(O)$, hence yields a hyperstructure $\mathcal{H} = (\mathcal{P}(O), \circ)$ in the sense assumed in the definition.)

Now define the hyperalgorithm $A : I \to \mathcal{P}(O)$ by

$$A(x) := \circ\big(\{A_1(x), A_2(x)\}\big).$$

Then, explicitly,

$$A(x) = \{x, -x\} \cup \{0\} = \{x, -x, 0\} \qquad (x \in \mathbb{Z}).$$

Hence the hyperalgorithm returns a *set* of admissible outputs that aggregates two algorithms and adds a default option 0.

The definition of a n-Superhyperalgorithm is provided below.

Definition 4.7.3 (n-Superhyperalgorithm)**.** Let I and O be nonempty sets. Suppose that $\mathcal{SH}_n = (\mathcal{P}_n(O), \circ^{(n)})$ is an n-superhyperstructure on O, where $\mathcal{P}_n(O)$ denotes the n-th powerset of O (constructed recursively as in the preliminaries) and $\circ^{(n)}$ is an n-superhyperoperation on $\mathcal{P}_n(O)$. Let

$$\mathcal{A} = \{A_i : I \to O \mid i \in J\}$$

be a finite collection of algorithms. An *n-Superhyperalgorithm* is a function

$$A^{(n)} : I \to \mathcal{P}_n(O)$$

defined by

$$A^{(n)}(x) = \circ^{(n)}\Big(\{A_i(x) : i \in J\}\Big) \quad \text{for all } x \in I.$$

This definition generalizes the hyperalgorithm by operating within the richer structure of the n-th powerset. In particular, when $n = 1$ (so that $\mathcal{P}_1(O) = \mathcal{P}(O)$), the n-superhyperalgorithm coincides with the hyperalgorithm.

Example 4.7.4 (A 2-superhyperalgorithm producing a nested family of candidate outputs)**.** Let $I = \mathbb{Z}$ and $O = \mathbb{Z}$, and keep the algorithms $A_1(x) = x$, $A_2(x) = -x$ from Example 4.7.2. Fix $n = 2$, so $\mathcal{P}_2(O) = \mathcal{P}(\mathcal{P}(O))$.

Define an n-level superhyperoperation $\circ^{(2)} : \mathcal{P}(O) \to \mathcal{P}_2(O)$ by

$$\circ^{(2)}(S) := \{\, S,\ S \cup \{0\} \,\} \qquad (S \subseteq O).$$

Thus $\circ^{(2)}(S)$ is a *set of subsets* of O, i.e. an element of $\mathcal{P}_2(O)$.

Define the 2-superhyperalgorithm $A^{(2)} : I \to \mathcal{P}_2(O)$ by

$$A^{(2)}(x) := \circ^{(2)}\big(\{A_1(x), A_2(x)\}\big).$$

Then for each $x \in \mathbb{Z}$,

$$A^{(2)}(x) = \Big\{\ \{x, -x\},\ \{x, -x, 0\}\ \Big\} \ \in\ \mathcal{P}_2(\mathbb{Z}).$$

Interpretation: the output is *nested*—it records two alternative candidate-sets (raw vs. augmented by a default 0), thereby illustrating how a superhyperalgorithm can encode hierarchical choice/uncertainty beyond a single hyperoutput set.

4.8 HyperFloorplan and SuperHyperFloorplan

Let us now build a *hyperfloorplan* starting from the set of modules S. We first consider the powerset $\mathcal{P}(S)$ [241]. Elements of $\mathcal{P}(S)$ are all possible subsets of modules. Our overarching goal is to capture geometric *feasibility* in a hyperoperation (cf. [242]).

Definition 4.8.1 (Hyperfloorplan)**.** (cf. [241, 242]) Let $S = \{M_1, \ldots, M_m\}$ be a finite set of rectangular modules, and let $\mathcal{P}(S)$ be its powerset. We say that a subset $X \subseteq S$ admits a *feasible floorplan* if there exists a classical slicing—or non-slicing—arrangement of the modules in X within some bounding rectangle satisfying all area, aspect-ratio, non-overlap, and connectivity constraints.

Define a hyperoperation

$$\circ \,:\, \mathcal{P}(S) \times \mathcal{P}(S) \;\longrightarrow\; \mathcal{P}\big(\mathcal{P}(S)\big)$$

by

$$A \circ B \;=\; \big\{\, X \subseteq S \mid A \cup B \subseteq X \text{ and } X \text{ admits a feasible floorplan} \big\}.$$

Then the *hyperfloorplan* of S is the hyperstructure

$$\mathcal{HF}(S) \;=\; \big(\mathcal{P}(S),\, \circ\big).$$

Example 4.8.2 (A hyperfloorplan on three modules)**.** Let $S = \{M_1, M_2, M_3\}$ be three rectangular modules. Assume the following (simplified) feasibility rule:

A subset $X \subseteq S$ admits a feasible floorplan if and only if $|X| \leq 2$.

This toy rule models a situation where any one or two modules can be placed without violating constraints, but placing all three simultaneously violates, e.g., aspect-ratio or wiring constraints.

For $A, B \subseteq S$, the hyperoperation in Definition 4.8.1 becomes

$$A \circ B = \big\{\, X \subseteq S \mid A \cup B \subseteq X \text{ and } |X| \leq 2 \,\big\}.$$

For instance, take $A = \{M_1\}$ and $B = \{M_2\}$. Then $A \cup B = \{M_1, M_2\}$, and the only superset $X \subseteq S$ with $|X| \leq 2$ is $X = \{M_1, M_2\}$. Hence

$$\{M_1\} \circ \{M_2\} = \big\{\{M_1, M_2\}\big\}.$$

On the other hand, if $A = \{M_1, M_2\}$ and $B = \{M_3\}$, then $A \cup B = S$ has size 3, so there is *no* feasible $X \subseteq S$ containing $A \cup B$ under the rule $|X| \leq 2$; therefore

$$\{M_1, M_2\} \circ \{M_3\} = \emptyset.$$

Thus $\mathcal{HF}(S) = (\mathcal{P}(S), \circ)$ is a concrete hyperfloorplan hyperstructure, whose outputs enumerate all feasible supersets extending a required module-set.

An n-superhyperfloorplan organizes hierarchical sets of modules using iterated powersets and feasibility constraints across n abstraction levels for multi-layer layout (cf. [241, 242]).

Definition 4.8.3 (n-Superhyperfloorplan)**.** (cf. [242]) Let $S = \{M_1, \ldots, M_m\}$ be a finite set of atomic modules, and define recursively the k-th powerset

$$\mathcal{P}_1(S) = \mathcal{P}(S), \qquad \mathcal{P}_{k+1}(S) = \mathcal{P}\big(\mathcal{P}_k(S)\big).$$

We now introduce a feasibility predicate F_k on subsets of $\mathcal{P}_k(S)$:

- $F_1(X)$ holds for $X \subseteq \mathcal{P}_1(S)$ if and only if there exists a classical (slicing or non-slicing) floorplan of the modules in X within some bounding rectangle, satisfying all area, aspect-ratio, non-overlap, and interconnection constraints.

- For $k > 1$, $F_k(X)$ holds for $X \subseteq \mathcal{P}_k(S)$ if and only if

  (i) There exists a bounding rectangle R and an arrangement of the "supermodules'' X inside R satisfying the usual floorplanning constraints at level k.

  (ii) Every element $Y \in X$, viewed as a subset $Y \subseteq \mathcal{P}_{k-1}(S)$, satisfies $F_{k-1}(Y)$.

For a fixed $n \geq 1$, define the hyperoperation

$$\circ_n : \underbrace{\mathcal{P}_n(S) \times \cdots \times \mathcal{P}_n(S)}_{m \text{ times}} \longrightarrow \mathcal{P}\big(\mathcal{P}_n(S)\big)$$

by

$$A_1 \circ_n A_2 \circ_n \cdots \circ_n A_m = \Big\{ X \in \mathcal{P}_n(S) \;\Big|\; \bigcup_{i=1}^{m} A_i \subseteq X \text{ and } F_n(X) \Big\}.$$

The *n-superhyperfloorplan* of S is the hyperstructure

$$\mathcal{SHF}_n = \big(\mathcal{P}_n(S), \circ_n\big).$$

Example 4.8.4 (A 2-superhyperfloorplan with supermodules)**.** Let $S = \{M_1, M_2, M_3\}$. Fix $n = 2$, so

$$\mathcal{P}_1(S) = \mathcal{P}(S), \qquad \mathcal{P}_2(S) = \mathcal{P}(\mathcal{P}(S)).$$

Elements of $\mathcal{P}_2(S)$ are *collections of module-subsets*, which we interpret as *supermodules* (clusters of clusters).

Adopt the following simplified hierarchical feasibility predicates:

- $F_1(X)$ holds for $X \subseteq \mathcal{P}_1(S)$ if and only if every $A \in X$ has $|A| \leq 2$. (This means: at the base level, each selected module-set can be floorplanned.)

- $F_2(\mathcal{X})$ holds for $\mathcal{X} \subseteq \mathcal{P}_2(S)$ if and only if each $Y \in \mathcal{X}$ satisfies $F_1(Y)$. (This means: each chosen supermodule is itself feasible at level 1.)

Now choose two level-2 inputs (two required supermodules)

$$A_1 = \big\{\{M_1\}, \{M_2\}\big\} \in \mathcal{P}_2(S), \qquad A_2 = \big\{\{M_2\}, \{M_3\}\big\} \in \mathcal{P}_2(S).$$

Take $m = 2$ (two inputs) in the definition of $\circ_n$. Then

$$A_1 \circ_2 A_2 = \Big\{ \mathcal{X} \in \mathcal{P}_2(S) \;\Big|\; A_1 \cup A_2 \subseteq \mathcal{X} \text{ and } F_2(\mathcal{X}) \Big\}.$$

A simple feasible choice is

$$\mathcal{X}_0 := A_1 \cup A_2 = \big\{\{M_1\}, \{M_2\}, \{M_3\}\big\} \in \mathcal{P}_2(S),$$

and indeed $F_2(\mathcal{X}_0)$ holds because every element of $\mathcal{X}_0$ is a singleton subset of S, hence satisfies the level-1 constraint $|A| \leq 2$. Therefore

$$\mathcal{X}_0 \in A_1 \circ_2 A_2,$$

so the 2-superhyperfloorplan output is nonempty. In words, $\mathcal{X}_0$ is a feasible level-2 supermodule collection that contains both required supermodules A_1 and A_2.

Thus $\mathcal{SHF}_2 = (\mathcal{P}_2(S), \circ_2)$ is a concrete 2-superhyperfloorplan hyperstructure, capturing feasibility constraints across two abstraction levels.

Chapter 5

SuperHyperUncertain Logic

In the literature, many uncertainty-aware logical frameworks are known, such as Fuzzy Logic, Intuitionistic Fuzzy Logic, Neutrosophic Logic, Plithogenic Logic, Four-valued Logic [243–245], and Many-valued Logic [246,247]. These frameworks can also be extended by using HyperStructures and SuperHyperStructures. In this chapter, we describe SuperHyperUncertain Logic. For reference, Table 5.1 summarizes classical uncertainty-aware set concepts and their hyper/superhyper counterparts.

5.1 SuperHyperFuzzy Set

The Fuzzy Set is a well-known concept used to address uncertainty in set theory [248–250]. In the field of fuzzy set theory, related concepts such as bipolar fuzzy sets [251,252], interval-valued fuzzy sets [253–255], m-polar fuzzy sets [256], spherical fuzzy sets [257, 258], Linear Diophantine fuzzy sets [259, 260], and hesitant fuzzy sets [261, 262] are also known. These sets can be extended into Hyperfuzzy Sets [263, 264] and SuperHyperfuzzy Sets [265] using hyperstructures and superhyperstructures. The definition is provided below .

Definition 5.1.1 (Fuzzy Set)**.** [248] A *fuzzy set* τ in a non-empty universe Y is a mapping $\tau : Y \to [0, 1]$. A *fuzzy relation* on Y is a fuzzy subset δ in $Y \times Y$. If τ is a fuzzy set in Y and δ is a fuzzy relation on Y, then δ is called a *fuzzy relation on* τ if

$$\delta(y, z) \leq \min\{\tau(y), \tau(z)\} \quad \text{for all } y, z \in Y.$$

Definition 5.1.2 (HyperFuzzy Set)**.** [6, 7, 19, 266] Let X be a non-empty set. A *hyperfuzzy set* over X is defined as a mapping:

$$\tilde{\mu} : X \to \mathcal{P}([0, 1]) \setminus \{\emptyset\},$$

where $\mathcal{P}([0, 1]) \setminus \{\emptyset\}$ represents the family of all non-empty subsets of the interval $[0, 1]$.

For each element $x \in X$, $\tilde{\mu}(x)$ assigns a non-empty subset of $[0, 1]$, representing the possible membership degrees of x in the hyperfuzzy set. This definition generalizes classical fuzzy sets by allowing the membership degree of each element to be a range (set of values) instead of a single scalar value.

Table 5.1: Classical uncertainty-aware set concepts and their hyper/superhyper counterparts.

Classical Concepts	**HyperConcepts**	**SuperHyperConcepts**
Fuzzy set	Hyperfuzzy set	SuperHyperFuzzy set
Neutrosophic set	Hyperneutrosophic set	SuperHyperNeutrosophic set
Plithogenic set	Hyperplithogenic set	SuperHyperPlithogenic set
Uncertain set	Hyperuncertain set	SuperHyperUncertain set
Soft set	Hypersoft set	SuperHyperSoft set
Rough set	Hyperrough set	SuperHyperRough set
Near set	Hypernear set	SuperHyperNear set
Weighted set	Hyperweighted set	SuperHyperWeighted set
Z-number	Hyper Z-number	SuperHyper Z-number
D-number	Hyper D-number	SuperHyper D-number
Grey set	Hypergrey set	SuperHyperGrey set

Example 5.1.3 (Hypercrowd-satisfaction as a HyperFuzzy Set)**.** Let

$$X = \{\mathrm{R1}, \mathrm{R2}, \mathrm{R3}\}$$

be three restaurants. A hyperfuzzy set $\tilde{\mu} : X \to \mathcal{P}([0,1]) \setminus \{\emptyset\}$ can model uncertainty/variability in customer satisfaction scores by assigning each restaurant a *set* of plausible membership degrees.

Define $\tilde{\mu}$ by

$$\tilde{\mu}(\mathrm{R1}) = [0.70, 0.90], \qquad \tilde{\mu}(\mathrm{R2}) = \{0.50, 0.60, 0.70\}, \qquad \tilde{\mu}(\mathrm{R3}) = [0.20, 0.40] \cup \{0.60\}.$$

Each $\tilde{\mu}(\mathrm{Ri})$ is a nonempty subset of $[0,1]$, so $\tilde{\mu}$ is a HyperFuzzy Set. Interpretation: R1 has consistently high satisfaction (an interval), R2 has three typical discrete rating levels, and R3 is mostly low but sometimes rated highly.

A (m,n)-Superhyperfuzzy Set maps m-level nested subsets of X to n-level nested fuzzy membership-value subsets in $[0,1]$, capturing distinct hierarchical fuzzy uncertainty patterns. The definition of the (m,n)-SuperhyperFuzzy Set is presented below.

Definition 5.1.4 ((m,n)-SuperhyperFuzzy Set)**.** Let U be a universe of discourse and let $A \subseteq U$ be nonempty. Fix integers $m, n \geq 0$. Define the m-th nested power-set of A by

$$\mathcal{P}^0(A) = A, \qquad \mathcal{P}^k(A) = \mathcal{P}\big(\mathcal{P}^{k-1}(A)\big) \quad (k \geq 1),$$

and similarly define $\mathcal{P}^n([0,1])$ for the unit interval. An (m,n)-*SuperhyperFuzzy Set* on A is a function

$$\tau : \mathcal{P}^m(A) \longrightarrow \mathcal{P}^n([0,1])$$

such that for each $X \in \mathcal{P}^m(A)$, the image $\tau(X) \subseteq [0,1]$ is nonempty. In other words, the "membership grade" of the (possibly nested) element X is not a single number but a set of values—intervals and/or discrete points—drawn from $[0,1]$.

Example 5.1.5 (A $(1,2)$-SuperhyperFuzzy Set on a small universe)**.** Let $U = \{a, b, c\}$ and take $A = U$. Choose $(m,n) = (1,2)$. Then

$$\mathcal{P}^1(A) = \mathcal{P}(A), \qquad \mathcal{P}^2([0,1]) = \mathcal{P}(\mathcal{P}([0,1])),$$

Table 5.2: A concise comparison of fuzzy, hyperfuzzy, and (m, n)–superhyperfuzzy sets.

	Fuzzy Set	**HyperFuzzy Set**	(m, n)–**SuperHyperFuzzy Set**
Universe / domain	Y $(y \in Y)$	X $(x \in X)$	$A \subseteq U$, inputs $X \in \mathcal{P}^m(A)$
Membership map (type)	$\tau : Y \to [0,1]$	$\tilde{\mu} : X \to \mathcal{P}([0,1]) \setminus \{\emptyset\}$	$\tau : \mathcal{P}^m(A) \to \mathcal{P}^n([0,1])$
Output meaning	single degree	nonempty *set* of degrees	n-level nested (nonempty) family of degree-sets
Uncertainty captured	point-valued uncertainty	one-level set-valued uncertainty	hierarchical / multi-level fuzzy uncertainty
Special-case relation	—	recovers fuzzy sets via singletons $\{r\}$	recovers fuzzy/hyperfuzzy via parameter choice and canonical embeddings
Typical example output	$\tau(y) = 0.7$	$\tilde{\mu}(x) = [0.6, 0.8]$ or $\{0.5, 0.7\}$	$\tau(X) = \{\{0.8\}, [0.75, 0.85]\}$ for $(m, n) = (1, 2)$

so τ assigns to each subset $X \subseteq A$ a *nonempty family of nonempty subsets of* $[0,1]$.

Define $\tau : \mathcal{P}(A) \to \mathcal{P}^2([0,1])$ on representative inputs by

$$\tau(\{a\}) = \Big\{\{0.8\},\ [0.75, 0.85]\Big\}, \qquad \tau(\{b\}) = \Big\{\{0.4, 0.5\}\Big\},$$

$$\tau(\{a, b\}) = \Big\{\ [0.60, 0.70],\ \{0.65\}\Big\}, \qquad \tau(\emptyset) = \Big\{\{0\}\Big\},$$

and similarly for the remaining subsets (any nonempty elements of $\mathcal{P}^2([0,1])$ may be chosen).

Each value $\tau(X)$ is a nonempty subset of $\mathcal{P}([0,1])$, hence $\tau(X) \in \mathcal{P}^2([0,1])$, and every "leaf" set of degrees is nonempty. Therefore τ is a concrete $(1,2)$-SuperhyperFuzzy Set. Interpretation: each $X \subseteq A$ has multiple admissible membership-degree sets (e.g. from different assessors or methods), capturing two-level (nested) fuzzy uncertainty.

For reference, we provide a concise comparison of fuzzy, hyperfuzzy, and (m, n)–superhyperfuzzy sets in Table 5.2.

5.2 SuperHyperNeutrosophic Set

A Neutrosophic Set models uncertainty using three membership functions: truth (T), indeterminacy (I), and falsity (F), which satisfy:

$$0 \leq T + I + F \leq 3.$$

[267,268]. As related concepts of neutrosophic sets, hesitant neutrosophic sets [269], bipolar neutrosophic sets [270–272], spherical neutrosophic sets [273, 274], and interval-valued neutrosophic sets [275–278] are also known. These sets can be extended into HyperNeutrosophic Sets [8] and SuperHyperNeutrosophic Sets using hyperstructures and superhyperstructures [279, 280].

Table 5.3: A concise comparison of fuzzy sets and neutrosophic sets.

	Fuzzy Set	**Neutrosophic Set**
Universe	Nonempty set X	Nonempty set X
Membership description	Single membership function $\mu_A : X \to [0,1]$	Three functions $T_A, I_A, F_A : X \to [0,1]$
Interpretation	Degree of membership (truth)	T=truth (presence), I=indeterminacy, F=falsity (absence)
Constraints	$0 \leq \mu_A(x) \leq 1$	$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$
Information captured	One scalar per element	Three components per element, explicitly representing uncertainty/indeterminacy
Special-case relation	—	Fuzzy sets are recovered e.g. by setting $I_A(x) = 0$ and $F_A(x) = 1 - T_A(x)$

Definition 5.2.1 (Neutrosophic Set)**.** [268, 281] Let X be a non-empty set. A *Neutrosophic Set (NS)* A on X is characterized by three membership functions:

$$T_A : X \to [0,1], \quad I_A : X \to [0,1], \quad F_A : X \to [0,1],$$

where for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degrees of truth, indeterminacy, and falsity, respectively. These values satisfy the following condition:

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3.$$

For reference, we provide a comparison of fuzzy sets and neutrosophic sets in Table 5.3.

Definition 5.2.2 (HyperNeutrosophic Set)**.** [19, 282] Let X be a non-empty set. A mapping $\tilde{\mu} : X \to \tilde{P}([0,1]^3)$ is called a *HyperNeutrosophic Set* over X, where $\tilde{P}([0,1]^3)$ denotes the family of all non-empty subsets of the unit cube $[0,1]^3$. For each $x \in X$, $\tilde{\mu}(x) \subseteq [0,1]^3$ represents a set of neutrosophic membership degrees, each consisting of truth (T), indeterminacy (I), and falsity (F) components, satisfying:

$$0 \leq T + I + F \leq 3.$$

Example 5.2.3 (HyperNeutrosophic medical assessments)**.** Neutrosophic sets are among the concepts widely used in fields such as medical assessments and medical analysis (cf. [283–286]). Let $X = \{\text{Alice}, \text{Bob}\}$ be two patients being screened for a disease. Define a mapping $\tilde{\mu} : X \to \tilde{\mathcal{P}}([0,1]^3)$ by assigning to each patient a nonempty set of possible neutrosophic triples (T, I, F), where T is truth (presence), I indeterminacy, and F falsity (absence).

Set

$$\tilde{\mu}(\text{Alice}) = \big\{(0.85, 0.10, 0.05),\ (0.80, 0.15, 0.05)\big\},$$

$$\tilde{\mu}(\text{Bob}) = \big\{(0.60, 0.25, 0.15),\ (0.65, 0.20, 0.10)\big\}.$$

Each triple satisfies $0 \leq T + I + F \leq 3$; e.g., $0.85 + 0.10 + 0.05 = 1.00$ and $0.60 + 0.25 + 0.15 = 1.00$. Hence $\tilde{\mu}$ is a HyperNeutrosophic Set on X in the sense of the definition. We present a schematic illustration of the concrete example in Fig. 5.1.

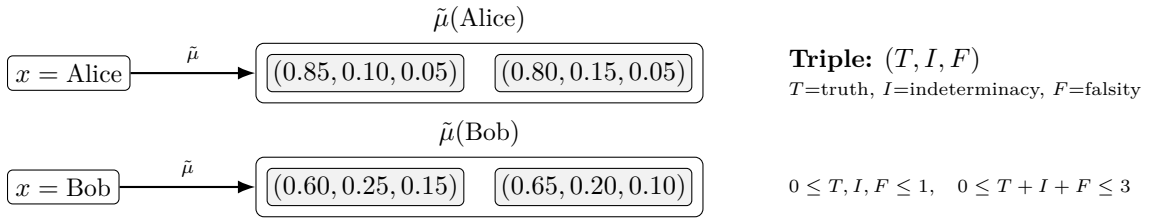


Figure 5.1: Schematic view of a HyperNeutrosophic set $\tilde{\mu} : X \to \tilde{\mathcal{P}}([0,1]^3)$ on $X = \{\text{Alice}, \text{Bob}\}$.

Definition 5.2.4 ((m,n)-SuperhyperNeutrosophic Set)**.** [287] Let U be a universe of discourse and let $A \subseteq U$ be nonempty. Fix $m, n \in \mathbb{N} \cup \{0\}$. Define recursively

$$\mathcal{P}^{\,0}(A) = A, \qquad \mathcal{P}^{\,k}(A) = \mathcal{P}\big(\mathcal{P}^{\,k-1}(A)\big) \quad (k \ge 1),$$

and analogously for $\mathcal{P}^{\,n}([0,1]^3)$. An (m,n)-*SuperhyperNeutrosophic Set* on A is a mapping

$$\mu : \ \mathcal{P}^{\,m}(A) \longrightarrow \mathcal{P}^{\,n}\big([0,1]^3\big)$$

such that for each $X \in \mathcal{P}^{\,m}(A)$ the value $\mu(X)$ is nonempty and every triple (T, I, F) occurring at level 1 inside $\mu(X)$ satisfies

$$0 \ \le \ T + I + F \ \le \ 3.$$

Equivalently, $\mathrm{U}_{n\to 1}\big(\mu(X)\big) \subseteq [0,1]^3$ is nonempty and consists of triples obeying $T + I + F \le 3$, where $\mathrm{U}_{n\to 1}$ denotes iterated union (flattening) from level n to level 1.

Example 5.2.5 (A $(1,2)$-SuperhyperNeutrosophic Set)**.** Let $U = \{a, b\}$ and take $A = U$. Choose $(m,n) = (1,2)$. Then $\mathcal{P}^1(A) = \mathcal{P}(A)$, and $\mathcal{P}^2([0,1]^3) = \mathcal{P}(\mathcal{P}([0,1]^3))$.

Define $\mu : \mathcal{P}(A) \to \mathcal{P}^2([0,1]^3)$ by specifying its values on representative inputs:

$$\mu(\{a\}) = \Big\{\{(0.90, 0.05, 0.05)\},\ \{(0.80, 0.15, 0.05), (0.75, 0.20, 0.05)\}\Big\},$$

$$\mu(\{b\}) = \Big\{\{(0.50, 0.30, 0.20)\}\Big\},$$

$$\mu(\{a, b\}) = \Big\{\{(0.70, 0.20, 0.10)\},\ \{(0.65, 0.25, 0.10)\}\Big\},$$

$$\mu(\emptyset) = \Big\{\{(0, 0, 0)\}\Big\}.$$

Each $\mu(X)$ is a nonempty set of nonempty subsets of $[0,1]^3$, hence $\mu(X) \in \mathcal{P}^2([0,1]^3)$. Moreover, every triple (T, I, F) appearing at the leaf level satisfies $0 \le T + I + F \le 3$ (e.g., $0.90 + 0.05 + 0.05 = 1.00$, $0.50 + 0.30 + 0.20 = 1.00$, etc.).

Thus μ is a concrete $(1,2)$-SuperhyperNeutrosophic Set on A: each subset $X \subseteq A$ is assigned a *family* of alternative neutrosophic assessments, encoding a second level of uncertainty (e.g., multiple experts or multiple diagnostic modalities).

5.3 SuperHyperPlithogenic Set

A Plithogenic Set is a mathematical framework that incorporates multi-valued degrees of appurtenance and contradictions, making it suitable for complex decision-making processes [288–293]. Various studies have been conducted on Plithogenic Sets [294, 295]. The definition is presented below.

Definition 5.3.1 (Plithogenic Set)**.** [292, 294] Let S be a universal set, and $P \subseteq S$. A *Plithogenic Set* PS is defined as:

$$PS = (P, v, Pv, pdf, pCF)$$

where:

- v is an attribute.
- Pv is the range of possible values for the attribute v.
- $pdf : P \times Pv \to [0,1]^s$ is the *Degree of Appurtenance Function (DAF).*
- $pCF : Pv \times Pv \to [0,1]^t$ is the *Degree of Contradiction Function (DCF).*

These functions satisfy the following axioms for all $a, b \in Pv$:

1. *Reflexivity of Contradiction Function*:

$$pCF(a, a) = 0$$

2. *Symmetry of Contradiction Function*:

$$pCF(a, b) = pCF(b, a)$$

Definition 5.3.2 (HyperPlithogenic Set)**.** [282, 296] Let X be a non-empty set, and let A be a set of attributes. For each attribute $v \in A$, let Pv be the set of possible values of v. A *HyperPlithogenic Set* HPS over X is defined as:

$$HPS = (P, \{v_i\}_{i=1}^n, \{Pv_i\}_{i=1}^n, \{\tilde{pdf}_i\}_{i=1}^n, pCF)$$

where:

- $P \subseteq X$ is a subset of the universe.

- For each attribute v_i, Pv_i is the set of possible values.

- For each attribute v_i, $\tilde{pdf}_i : P \times Pv_i \to \tilde{P}([0,1]^s)$ is the *Hyper Degree of Appurtenance Function (HDAF)*, assigning to each element $x \in P$ and attribute value $a_i \in Pv_i$ a set of membership degrees.

- $pCF : (\bigcup_{i=1}^{n} Pv_i) \times (\bigcup_{i=1}^{n} Pv_i) \to [0,1]^t$ is the *Degree of Contradiction Function (DCF).*

Example 5.3.3 (HyperPlithogenic evaluation of smartphones)**.** Fuzzy sets, neutrosophic sets, and plithogenic sets have been used in applications such as the evaluation of smartphones and PCs [297–299]. Let

$$X = \{\mathrm{A}, \mathrm{B}, \mathrm{C}\}$$

be three smartphone models and set $P = X$. Consider two attributes:

$$v_1 = \text{Battery Life}, \quad P_{v_1} = \{\text{Short}, \text{Moderate}, \text{Long}\},$$

$$v_2 = \text{Camera Quality}, \quad P_{v_2} = \{\text{Low}, \text{Medium}, \text{High}\}.$$

Take $s = 1$ (scalar degrees), so $\tilde{P}([0,1]^s) = \tilde{P}([0,1])$.

Define the Hyper Degree of Appurtenance Functions $\tilde{pdf}_1 : P \times P_{v_1} \to \tilde{P}([0,1])$ and $\tilde{pdf}_2 : P \times P_{v_2} \to \tilde{P}([0,1])$ by, for example:

$$\tilde{pdf}_1(\mathrm{A}, \text{Long}) = \{0.70, 0.75\},$$

$$\tilde{pdf}_1(\mathrm{A}, \text{Moderate}) = [0.50, 0.60],$$

$$\tilde{pdf}_1(\mathrm{B}, \text{Long}) = [0.80, 0.90],$$

$$\tilde{pdf}_1(\mathrm{C}, \text{Short}) = \{0.30, 0.40\},$$

and

$$\tilde{pdf}_2(\mathrm{A}, \text{High}) = [0.65, 0.80],$$

$$\tilde{pdf}_2(\mathrm{B}, \text{High}) = \{0.85\},$$

$$\tilde{pdf}_2(\mathrm{C}, \text{Medium}) = \{0.55, 0.60\}.$$

Each value is a nonempty subset of $[0,1]$, hence lies in $\tilde{P}([0,1])$.

Define the Degree of Contradiction Function $pCF : (P_{v_1} \cup P_{v_2})^2 \to [0,1]$ (take $t = 1$) by

$$pCF(a, a) = 0 \ \text{ for all } a,$$

$$pCF(\text{Short}, \text{Long}) = 0.9,$$

$$pCF(\text{Low}, \text{High}) = 0.8,$$

and extend symmetrically: $pCF(a, b) = pCF(b, a)$.

Then

$$HPS = \big(P, \{v_1, v_2\}, \{P_{v_1}, P_{v_2}\}, \{\tilde{pdf}_1, \tilde{pdf}_2\}, pCF\big)$$

is a concrete HyperPlithogenic Set over X: each (x, a) is assigned a *set* of possible degrees, and contradictions between attribute values are quantified by pCF.

A (m,n)-SuperhyperPlithogenic Set assigns m-level parameter-subsets and their attribute values to n-level fuzzy-contradiction degree-sets, capturing multi-faceted membership and inter-attribute conflicts and uncertainty patterns. The definition of the (m,n)-SuperhyperPlithogenic Set is presented below.

Definition 5.3.4 ((m,n)-SuperhyperPlithogenic Set)**.** Let X be a nonempty set and let $V = \{v_1, \ldots, v_k\}$ be a finite set of attributes. For each $v \in V$, let P_v be the set of its possible values. Fix positive integers m, n and positive dimensions s, t. Define the m-th nested powerset of X by

$$\mathcal{P}^0(X) = X, \quad \mathcal{P}^r(X) = \mathcal{P}\big(\mathcal{P}^{r-1}(X)\big) \quad (r \geq 1),$$

and similarly $\mathcal{P}^n\big([0,1]^s\big)$ for the s-dimensional unit cube. An (m,n)-*SuperhyperPlithogenic Set* over X is the quintuple

$$SHP^{(m,n)} = \Big(\mathcal{P}^m(X),\ V,\ \{P_v\}_{v\in V},\ \{\tilde{pdf}_v^{(m,n)}\}_{v\in V},\ pCF^{(m,n)}\Big),$$

where

(i) $\mathcal{P}^m(X)$ is the domain of "super-elements" of level m.

(ii) For each $v \in V$, P_v is the finite set of its values.

(iii) The *Hyper Degree of Appurtenance Function*

$$\tilde{pdf}_v^{(m,n)} : \mathcal{P}^m(X) \times P_v \longrightarrow \mathcal{P}^n\big([0,1]^s\big)$$

assigns to each (A,a) with $A \in \mathcal{P}^m(X)$ and $a \in P_v$ a nonempty subset $\tilde{pdf}_v^{(m,n)}(A,a) \subseteq [0,1]^s$ representing all possible membership-degree vectors of dimension s.

(iv) The *Degree of Contradiction Function*

$$pCF^{(m,n)} : \Big(\bigcup_{v\in V} P_v\Big) \times \Big(\bigcup_{v\in V} P_v\Big) \longrightarrow [0,1]^t$$

satisfies for all a, b:

(a) $pCF^{(m,n)}(a,a) = 0$ (reflexivity),

(b) $pCF^{(m,n)}(a,b) = pCF^{(m,n)}(b,a)$ (symmetry).

Example 5.3.5 (A $(2,2)$-SuperhyperPlithogenic Set)**.** Let $X = \{\mathrm{A}, \mathrm{B}\}$ be two projects under evaluation and take $m = n = 2$. Then

$$\mathcal{P}^1(X) = \mathcal{P}(X),$$

$$\mathcal{P}^2(X) = \mathcal{P}(\mathcal{P}(X)).$$

Choose two attributes

$$v_1 = \text{Risk}, \quad P_{v_1} = \{\text{Low}, \text{High}\},$$

$$v_2 = \text{Cost}, \quad P_{v_2} = \{\text{Cheap}, \text{Expensive}\}.$$

Let $s = t = 1$, so degrees are scalars and contradiction values lie in $[0, 1]$, and $\mathcal{P}^2([0, 1]) = \mathcal{P}(\mathcal{P}([0, 1]))$.

A level-2 super-element is a subset $A \in \mathcal{P}^2(X)$. Fix

$$A := \big\{\{\text{A}\}, \{\text{A}, \text{B}\}\big\} \in \mathcal{P}^2(X).$$

Define hyper degrees of appurtenance $\tilde{pdf}_v^{(2,2)} : \mathcal{P}^2(X) \times P_v \to \mathcal{P}^2([0, 1])$ by specifying, for example,

$$\tilde{pdf}_{\text{Risk}}^{(2,2)}(A, \text{High}) = \big\{\{0.70, 0.75\},\ [0.72, 0.78]\big\},$$

$$\tilde{pdf}_{\text{Risk}}^{(2,2)}(A, \text{Low}) = \big\{\{0.30\}\big\},$$

$$\tilde{pdf}_{\text{Cost}}^{(2,2)}(A, \text{Cheap}) = \big\{\{0.60, 0.65\}\big\},$$

$$\tilde{pdf}_{\text{Cost}}^{(2,2)}(A, \text{Expensive}) = \big\{\{0.80\},\ [0.78, 0.85]\big\}.$$

Each value is a nonempty set of nonempty subsets of $[0, 1]$, hence an element of $\mathcal{P}^2([0, 1])$.

Define the contradiction function $pCF^{(2,2)} : (P_{v_1} \cup P_{v_2})^2 \to [0, 1]$ by

$$pCF^{(2,2)}(a, a) = 0, \qquad pCF^{(2,2)}(\text{Low}, \text{High}) = 0.9,$$

$$pCF^{(2,2)}(\text{Cheap}, \text{Expensive}) = 0.8,$$

with symmetry $pCF^{(2,2)}(a, b) = pCF^{(2,2)}(b, a)$.

Thus the quintuple

$$SHP^{(2,2)} = \Big(\mathcal{P}^2(X),\ \{v_1, v_2\},\ \{P_{v_1}, P_{v_2}\},\ \{\tilde{pdf}_{v_1}^{(2,2)}, \tilde{pdf}_{v_2}^{(2,2)}\},\ pCF^{(2,2)}\Big)$$

is a concrete $(2, 2)$-SuperhyperPlithogenic Set: its domain elements are level-2 nested subsets of X, its membership values are level-2 nested degree-sets in $[0, 1]$, and contradictions are quantified by $pCF^{(2,2)}$.

5.4 SuperHyperUncertain Set

An Uncertain Set maps each element to an admissible uncertainty degree in a fixed domain, unifying fuzzy, intuitionistic, neutrosophic, and plithogenic semantics [300]. A HyperUncertain Set assigns each element a nested family of admissible degree-tuples via iterated powersets, capturing grouped or multi-source uncertainty [19]. A SuperHyperUncertain Set maps higher-level powerset objects to nested families of admissible degree-tuples, combining hierarchical inputs with hierarchical uncertainty outputs [19].

Definition 5.4.1 (Uncertain Set)**.** [300] Let X be a nonempty universe of discourse. Fix an integer $r \geq 1$ and a set $D \subseteq [0,1]^r$ (the *admissible degree domain*), typically specified by constraints (e.g., $D = [0,1]$ for fuzzy sets; $D = \{(\mu, \nu) \in [0,1]^2 \mid \mu + \nu \leq 1\}$ for intuitionistic fuzzy sets; etc.).

An *uncertain set (of type D)* on X is a mapping

$$\mu : X \longrightarrow D \subseteq [0,1]^r.$$

For each $x \in X$, the value $\mu(x) \in D$ is interpreted as an r-tuple of membership degrees assigned to x.

Definition 5.4.2 (HyperUncertain Set)**.** [300] Let X be a nonempty universe, fix integers $r \geq 1$ and $n \geq 1$, and fix a degree domain $D \subseteq [0,1]^r$.

A (level-n) *hyperuncertain set* on X (with degree domain D) is a mapping

$$\widetilde{\mu} : X \longrightarrow \mathcal{P}^n(D) \quad (\text{or, if desired, } \widetilde{\mu} : X \to \mathcal{P}^n_*(D))$$

such that, for every $x \in X$, the flattened value satisfies

$$\mathrm{Flat}_n\big(\widetilde{\mu}(x)\big) \;\subseteq\; D.$$

Interpretation: $\widetilde{\mu}(x)$ is an n-level *nested family* of admissible degree-tuples, encoding "sets of candidate degrees", possibly with higher-level grouping (scenarios, sources, experts, etc.). The classical hyperuncertain case is $n = 1$, where $\widetilde{\mu}(x) \subseteq D$ is simply a (possibly non-singleton) set of admissible degree-tuples.

Definition 5.4.3 (SuperHyperUncertain Set)**.** [300] Let X be a nonempty base set. Fix integers $m \geq 1$ (the *super level*), $n \geq 1$ (the *hyper level*), and $r \geq 1$. Fix a degree domain $D \subseteq [0,1]^r$.

A (m,n)-*superhyperuncertain set* on X with degree domain D is a mapping

$$\mu^{(m,n)} : \mathcal{P}^m(X) \longrightarrow \mathcal{P}^n(D) \quad (\text{or, if desired, } \mu^{(m,n)} : \mathcal{P}^m(X) \to \mathcal{P}^n_*(D))$$

such that, for every $A \in \mathcal{P}^m(X)$,

$$\mathrm{Flat}_n\big(\mu^{(m,n)}(A)\big) \;\subseteq\; D.$$

5.5 SuperHyperSoft Set

A soft set maps each parameter to a subset of the universe, representing approximate descriptions without requiring membership grades explicitly [301–303]. A hypersoft set maps tuples of attribute-values to universe subsets, capturing multi-attribute parameterization and uncertainty across combined descriptors systematically overall [288, 304–306]. Related notions such as *fuzzy hypersoft sets* [307–309], hypersoft expert sets [310–312], and *neutrosophic hypersoft sets* [313–315] are also known. A superhypersoft set maps nested parameter-collections to nested universe subsets via iterated powersets, enabling hierarchical, multi-level soft modeling under uncertainty [316–318]. As related notions to *superhypersoft sets*, concepts such as *fuzzy superhypersoft sets* [318] and *neutrosophic superhypersoft sets* [317, 319, 320] are also known. It is also known that a SuperHyperSoft Set can be extended to an (m,n)-SuperHyperSoft set [321].

Definition 5.5.1 ((m,n)-SuperhyperSoft Set)**.** Let U be a nonempty universe and let S be a nonempty set of parameters. Fix integers $m, n \geq 1$. Using the already-defined nonempty iterated powersets $\tilde{\mathcal{P}}_m(S)$ and $\tilde{\mathcal{P}}_n(U)$, an (m,n)-*SuperhyperSoft Set* over U (with respect to S) is a pair

$$(F, \tilde{\mathcal{P}}_m(S)),$$

where

$$F : \tilde{\mathcal{P}}_m(S) \longrightarrow \tilde{\mathcal{P}}_n(U).$$

Equivalently, it is a mapping that assigns to each m-level (nested) nonempty parameter $\Gamma \in \tilde{\mathcal{P}}_m(S)$ an n-level (nested) nonempty subset $F(\Gamma) \in \tilde{\mathcal{P}}_n(U)$.

Example 5.5.2 (Real-world example: hierarchical procurement shortlists as a $(2,2)$-SuperhyperSoft Set)**.** *Scenario.* A company wants to procure a laptop model for its employees. Different departments (IT, Security, Finance) evaluate devices using *groups of criteria* and may produce *multiple shortlists* (e.g., per vendor, per region, or per risk level). This naturally leads to nested parameters and nested outputs.

Universe and parameters. Let the universe of available laptop models be

$$U = \{\mathrm{L}_1, \mathrm{L}_2, \mathrm{L}_3, \mathrm{L}_4\},$$

and let the parameter set be

$$S = \{\mathrm{Cost}, \mathrm{Performance}, \mathrm{Battery}, \mathrm{Weight}, \mathrm{Security}, \mathrm{Support}\}.$$

Fix $m = n = 2$. Thus $\tilde{\mathcal{P}}_2(S)$ consists of nonempty collections of nonempty subsets of S (groups of criteria), and $\tilde{\mathcal{P}}_2(U)$ consists of nonempty collections of nonempty subsets of U (collections of candidate shortlists).

A $(2,2)$-SuperhyperSoft Set. Define $F : \tilde{\mathcal{P}}_2(S) \to \tilde{\mathcal{P}}_2(U)$ by interpreting each $\Gamma \in \tilde{\mathcal{P}}_2(S)$ as a *nested parameter request* and $F(\Gamma)$ as the resulting *nested family of feasible shortlists.* For instance, set

$$\Gamma_1 := \big\{\{\mathrm{Cost}, \mathrm{Support}\},\ \{\mathrm{Security}\}\big\} \in \tilde{\mathcal{P}}_2(S),$$

meaning "optimize cost/support and satisfy security constraints" (two criterion groups). Then let

$$F(\Gamma_1) := \big\{\{\mathrm{L}_1, \mathrm{L}_2\},\ \{\mathrm{L}_2, \mathrm{L}_3\}\big\} \in \tilde{\mathcal{P}}_2(U),$$

where the two inner subsets represent, e.g., *Finance-approved* and *Security-approved* shortlists.

Similarly, define

$$\Gamma_2 := \big\{\{\mathrm{Performance}, \mathrm{Battery}\},\ \{\mathrm{Weight}\}\big\} \in \tilde{\mathcal{P}}_2(S),$$

and set

$$F(\Gamma_2) := \big\{\{\mathrm{L}_2, \mathrm{L}_4\},\ \{\mathrm{L}_4\}\big\} \in \tilde{\mathcal{P}}_2(U),$$

where the first shortlist may correspond to "engineering workflow" needs and the second to "travel-heavy staff" needs.

Interpretation. Each 2-level parameter object Γ (a collection of criterion bundles) is mapped to a 2-level nonempty collection $F(\Gamma)$ of nonempty candidate sets in U, capturing hierarchical and multi-stakeholder uncertainty. Hence $(F, \tilde{\mathcal{P}}_2(S))$ is a concrete $(2,2)$-SuperhyperSoft Set over U with respect to S.

5.6 SuperHyperRough Set

Rough set represents a subset via lower and upper approximations induced by an equivalence relation, capturing boundary-region uncertainty explicitly formally [322, 323]. HyperRough set maps attribute-value combinations to subsets of X, then computes Pawlak-style lower and upper approximations per parameter individually thereafter [324, 325]. SuperHyperRough set lifts parameters and outputs to iterated powersets, producing nested candidate subsets whose flattenings yield rough approximations levelwise systematically [321].

Definition 5.6.1 ((m,n)-SuperhyperRough Set)**.** [321] Let X be a nonempty finite universe and let $R \subseteq X \times X$ be an equivalence relation. Let J be an attribute (parameter) domain (e.g. a Cartesian product of attribute-value sets, as fixed earlier). Fix integers $m, n \geq 1$. An (m,n)-*SuperhyperRough Set* (based on R) is a mapping

$$F : \tilde{\mathcal{P}}_m(J) \longrightarrow \tilde{\mathcal{P}}_n(X),$$

together with the induced *flattening* map

$$\mathrm{Flat}_1(A) := A \quad (A \in \tilde{\mathcal{P}}_1(X)), \qquad \mathrm{Flat}_{t+1}(\mathcal{A}) := \bigcup_{B \in \mathcal{A}} \mathrm{Flat}_t(B) \quad (\mathcal{A} \in \tilde{\mathcal{P}}_{t+1}(X)),$$

so that $\mathrm{Flat}_n(W) \subseteq X$ is the underlying (nonempty) subset represented by $W \in \tilde{\mathcal{P}}_n(X)$.

For each $\Gamma \in \tilde{\mathcal{P}}_m(J)$, put

$$W_\Gamma := \mathrm{Flat}_n\big(F(\Gamma)\big) \subseteq X,$$

and define the lower and upper approximations (with respect to R) by

$$\underline{W_\Gamma} := \{\, x \in X \mid [x]_R \subseteq W_\Gamma \,\}, \qquad \overline{W_\Gamma} := \{\, x \in X \mid [x]_R \cap W_\Gamma \neq \varnothing \,\}.$$

The pair $\big(\underline{W_\Gamma}, \overline{W_\Gamma}\big)$ is the rough approximation associated with the super-parameter Γ.

Example 5.6.2 (Real-world example: cohort selection under partial observability in clinical triage)**.** *Universe and indiscernibility.* Let X be a finite set of patients in an emergency department:

$$X = \{p_1, p_2, p_3, p_4, p_5, p_6\}.$$

Assume we only observe a coarse symptom profile, so patients are *indiscernible* when they share that profile. Model this by an equivalence relation R whose equivalence classes are

$$[p_1]_R = [p_2]_R = \{p_1, p_2\}, \qquad [p_3]_R = [p_4]_R = \{p_3, p_4\}, \qquad [p_5]_R = [p_6]_R = \{p_5, p_6\}.$$

Attribute domain. Let J be a finite attribute/criterion domain used for triage decisions, e.g.

$$J = \{\text{Fever}, \text{Cough}, \text{Dyspnea}, \text{Diabetes}\}.$$

Fix $(m,n) = (2,2)$. Hence $\tilde{\mathcal{P}}_2(J)$ consists of nonempty collections of nonempty criterion-bundles, and $\tilde{\mathcal{P}}_2(X)$ consists of nonempty collections of nonempty patient-subsets.

A concrete $(2,2)$-SuperhyperRough Set. Define a mapping $F : \tilde{\mathcal{P}}_2(J) \to \tilde{\mathcal{P}}_2(X)$ by interpreting each $\Gamma \in \tilde{\mathcal{P}}_2(J)$ as a nested triage request (groups of criteria), and $F(\Gamma)$ as a nested family of candidate patient sets returned by different rules/teams.

For instance, let

$$\Gamma := \big\{\{\text{Fever}, \text{Cough}\},\ \{\text{Diabetes}\}\big\} \in \tilde{\mathcal{P}}_2(J),$$

meaning: "respiratory-symptom group" together with a "comorbidity group". Assume two triage sources produce the candidate subsets

$$F(\Gamma) := \big\{\{p_1, p_3, p_5\},\ \{p_2, p_3\}\big\} \in \tilde{\mathcal{P}}_2(X).$$

Flattening and rough approximations. By flattening at level $n = 2$,

$$W_\Gamma = \mathrm{Flat}_2\big(F(\Gamma)\big) = \{p_1, p_3, p_5\} \cup \{p_2, p_3\} = \{p_1, p_2, p_3, p_5\} \subseteq X.$$

The lower approximation (certain membership) is

$$\underline{W_\Gamma} = \{x \in X \mid [x]_R \subseteq W_\Gamma\} = \{p_1, p_2\},$$

because $\{p_1, p_2\} \subseteq W_\Gamma$ while $\{p_3, p_4\} \nsubseteq W_\Gamma$ and $\{p_5, p_6\} \nsubseteq W_\Gamma$. The upper approximation (possible membership) is

$$\overline{W_\Gamma} = \{x \in X \mid [x]_R \cap W_\Gamma \neq \varnothing\} = \{p_1, p_2, p_3, p_4, p_5, p_6\} = X,$$

since every class $\{p_1, p_2\}$, $\{p_3, p_4\}$, and $\{p_5, p_6\}$ intersects W_Γ.

Interpretation. Here F encodes a hierarchical (multi-source) candidate-cohort construction from grouped criteria Γ, while $(\underline{W_\Gamma}, \overline{W_\Gamma})$ quantifies what can be *certainly* vs. *possibly* selected under the coarse observational equivalence R.

5.7 SuperHyperNear Set

A near set groups objects considered close when their feature descriptions overlap, using descriptive proximity rather than metric distance alone [326–328]. A hypernear set compares object-subsets via probe-feature images; nearness holds when closures intersect nontrivially within feature space under chosen probes. An (m, n)-SuperHyperNear set treats n-level nested subsets; apply m probes, flatten features, and test self-nearness by closure-intersection in Euclidean space.

Definition 5.7.1 ((m, n)-SuperHyperNear Set)**.** Let X be a nonempty (finite) set and fix an integer $m \geq 1$. Let $\varphi_1, \ldots, \varphi_m : X \to \mathbb{R}$ be probe functions and set the feature map

$$\Phi : X \longrightarrow \mathbb{R}^m, \qquad \Phi(x) = \big(\varphi_1(x), \ldots, \varphi_m(x)\big).$$

Fix $n \geq 1$. Define the level-n extended feature map $\Phi_e^{(n)} : P_n(X) \to \mathcal{P}(\mathbb{R}^m)$ recursively by

$$\Phi_e^{(1)}(A) = \{\Phi(a) \mid a \in A\} \quad (A \in P_1(X) = \mathcal{P}(X)),$$

and for $n \geq 2$,

$$\Phi_e^{(n)}(\mathcal{A}) = \bigcup_{A \in \mathcal{A}} \Phi_e^{(n-1)}(A), \qquad \mathcal{A} \in P_n(X) \ \ (\text{so } \mathcal{A} \subseteq P_{n-1}(X)).$$

(Equivalently, $\Phi_e^{(n)}(\cdot)$ is obtained by iterating "apply Φ at the leaves and then flatten by unions".)

Define the superhyper-nearness relation $\delta_{\Phi}^{(n)}$ on $P_n(X)$ by

$$\mathcal{A}\ \delta_{\Phi}^{(n)}\ \mathcal{B} \quad \Longleftrightarrow \quad \overline{\Phi_e^{(n)}(\mathcal{A})}\ \cap\ \overline{\Phi_e^{(n)}(\mathcal{B})}\ \neq\ \emptyset,$$

where closure is taken in the usual Euclidean topology on $\mathbb{R}^m$.

An element $\mathcal{A} \in P_n(X)$ is called an (m,n)-*SuperHyperNear Set* (with respect to Φ) if

$$\mathcal{A}\ \delta_{\Phi}^{(n)}\ \mathcal{A}.$$

Example 5.7.2 (Real-world example: near plan-families in a sensor network via shared feature signatures)**.** *Scenario.* Let X be a finite set of IoT sensors deployed in a building. Each sensor $x \in X$ has a measured temperature and humidity. We regard two *collections of sensor-groups* as *near* if, after mapping all sensors to their feature vectors and flattening, the resulting feature point-sets intersect.

Data and probes. Let

$$X = \{s_1, s_2, s_3, s_4\}, \qquad m = 2,$$

and define probe functions $\varphi_1, \varphi_2 : X \to \mathbb{R}$ by

$$\varphi_1(x) = \text{temperature of } x\ (^\circ\text{C}), \qquad \varphi_2(x) = \text{humidity of } x\ (\%).$$

Assume the (snapshot) readings are

$$\Phi(s_1) = (22, 40), \quad \Phi(s_2) = (23, 42), \quad \Phi(s_3) = (19, 55), \quad \Phi(s_4) = (24, 38),$$

where $\Phi(x) = (\varphi_1(x), \varphi_2(x)) \in \mathbb{R}^2$.

Choose $n = 2$ and two elements of $P_2(X)$. Recall $P_2(X) = \mathcal{P}(\mathcal{P}(X))$. Consider

$$\mathcal{A} := \big\{\{s_1, s_2\}, \{s_3\}\big\} \in P_2(X), \qquad \mathcal{B} := \big\{\{s_2\}, \{s_4\}\big\} \in P_2(X).$$

Interpretation: $\mathcal{A}$ and $\mathcal{B}$ are two *plan-families* (collections) of candidate sensor-groups.

Extended feature map and flattening. By definition,

$$\Phi_e^{(1)}(A) = \{\Phi(a) \mid a \in A\} \quad (A \subseteq X), \qquad \Phi_e^{(2)}(\mathcal{C}) = \bigcup_{A \in \mathcal{C}} \Phi_e^{(1)}(A) \quad (\mathcal{C} \in P_2(X)).$$

Hence

$$\Phi_e^{(2)}(\mathcal{A}) = \Phi_e^{(1)}(\{s_1, s_2\}) \cup \Phi_e^{(1)}(\{s_3\}) = \{(22, 40), (23, 42)\} \cup \{(19, 55)\},$$

so

$$\Phi_e^{(2)}(\mathcal{A}) = \{(22, 40), (23, 42), (19, 55)\}.$$

Similarly,

$$\Phi_e^{(2)}(\mathcal{B}) = \Phi_e^{(1)}(\{s_2\}) \cup \Phi_e^{(1)}(\{s_4\}) = \{(23, 42)\} \cup \{(24, 38)\} = \{(23, 42), (24, 38)\}.$$

Superhyper-nearness. Since these images are finite subsets of $\mathbb{R}^2$, they are closed in the Euclidean topology; thus $\overline{\Phi_e^{(2)}(\mathcal{A})} = \Phi_e^{(2)}(\mathcal{A})$ and $\overline{\Phi_e^{(2)}(\mathcal{B})} = \Phi_e^{(2)}(\mathcal{B})$. Therefore

$$\overline{\Phi_e^{(2)}(\mathcal{A})} \cap \overline{\Phi_e^{(2)}(\mathcal{B})} = \{(22, 40), (23, 42), (19, 55)\} \cap \{(23, 42), (24, 38)\} = \{(23, 42)\} \neq \varnothing,$$

so $\mathcal{A}\ \delta_{\Phi}^{(2)}\ \mathcal{B}$.

Moreover, $\mathcal{A}\ \delta_{\Phi}^{(2)}\ \mathcal{A}$ holds because $\overline{\Phi_e^{(2)}(\mathcal{A})} \cap \overline{\Phi_e^{(2)}(\mathcal{A})} = \overline{\Phi_e^{(2)}(\mathcal{A})} \neq \varnothing$. Hence $\mathcal{A}$ is an $(m,n) = (2,2)$-*SuperHyperNear Set* (with respect to Φ).

5.8 SuperHyperWeighted Set

A weighted set assigns each element a nonnegative scalar weight, quantifying its relative importance or frequency in analyses and computations (cf. [329–331]). A hyperweighted set maps each element to a nonempty set of scalar weights, representing multiple uncertain valuation or importance scenarios [282, 332, 333]. An n-superhyperweighted set assigns each element a nonempty nested collection of depth n+1 weight families capturing hierarchical uncertainty and dependencies [282].

Definition 5.8.1 (Weighted Set)**.** (cf. [329–331]) A *Weighted Set* is an ordered pair (S, w) where:

- S is a non-empty set.
- $w : S \to \mathbb{R}$ is a function that assigns a unique real number to each element $s \in S$; that is, for every $s \in S$, there exists a uniquely determined weight $w(s) \in \mathbb{R}$.

Definition 5.8.2 (Hyperweighted Set)**.** [282,332] Let S be a non-empty set. A *Hyperweighted Set* is a pair (S, W) where

$$W : S \to \mathcal{P}(\mathbb{R}) \setminus \{\emptyset\}$$

is a function that assigns to each element $s \in S$ a non-empty subset $W(s) \subseteq \mathbb{R}$. In other words, every element s is associated with multiple weights.

Example 5.8.3 (Hyperweighted set of product prices)**.** Let $S = \{\mathrm{P}_1, \mathrm{P}_2, \mathrm{P}_3\}$ be three products. Define a weight assignment $W : S \to \mathcal{P}(\mathbb{R}) \setminus \{\emptyset\}$ by

$$W(\mathrm{P}_1) = \{98, 100, 105\}, \qquad W(\mathrm{P}_2) = [49, 55], \qquad W(\mathrm{P}_3) = \{120\}.$$

Each $W(\mathrm{P}_i)$ is a nonempty subset of $\mathbb{R}$, interpreted as plausible prices (e.g., from different vendors or time windows). Hence (S, W) is a Hyperweighted Set.

Definition 5.8.4 (n-Superhyperweighted Set)**.** (cf. [282]) Let S be a non-empty set and let $n \geq 0$ be an integer. An *n-Superhyperweighted Set* is an ordered pair

$$(S, \mathcal{W}_n),$$

where the function

$$\mathcal{W}_n : S \to P_{n+1}(\mathbb{R}) \setminus \{\varnothing\}$$

assigns to each element $s \in S$ a nonempty nested collection of weights. In particular:

- For $n = 0$: $\mathcal{W}_0(s) \in P_1(\mathbb{R}) = \mathcal{P}(\mathbb{R})$ represents a nonempty set of real numbers (a standard Hyperweighted Set).
- For $n = 1$: $\mathcal{W}_1(s) \in P_2(\mathbb{R}) = \mathcal{P}(\mathcal{P}(\mathbb{R}))$ represents a collection of nonempty subsets of $\mathbb{R}$ (a Superhyperweighted Set).

- For $n = 2$: $\mathcal{W}_2(s) \in P_3(\mathbb{R})$ represents a nested structure with three layers.

Example 5.8.5 (A 1-Superhyperweighted set (nested weight families))**.** Let $S = \{\mathrm{A}, \mathrm{B}\}$. Take $n = 1$, so the codomain is

$$P_{n+1}(\mathbb{R}) = P_2(\mathbb{R}) = \mathcal{P}(\mathcal{P}(\mathbb{R})).$$

Define $\mathcal{W}_1 : S \to P_2(\mathbb{R}) \setminus \{\emptyset\}$ by

$$\mathcal{W}_1(\mathrm{A}) = \big\{\{1.0, 1.2\},\ [0.9, 1.1]\big\}, \qquad \mathcal{W}_1(\mathrm{B}) = \big\{\{2.0\},\ \{1.8, 1.9, 2.1\}\big\}.$$

Each $\mathcal{W}_1(s)$ is a nonempty *collection of nonempty subsets* of $\mathbb{R}$, so $(S, \mathcal{W}_1)$ is a 1-Superhyperweighted Set. Interpretation: each element has multiple "weight-scenarios", and each scenario is itself a set (or interval) of weights.

Definition 5.8.6 ((m, n)-Superhyperweighted Set)**.** Let S be a nonempty set and let $m, n \in \mathbb{N}_0$. Let $P_t(\mathbb{R})$ denote the iterated powersets of $\mathbb{R}$, with $P_0(\mathbb{R}) = \mathbb{R}$ and $P_{t+1}(\mathbb{R}) = \mathcal{P}(P_t(\mathbb{R}))$ for $t \geq 0$. Let $P_t^*(\mathbb{R}) := P_t(\mathbb{R}) \setminus \{\emptyset\}$ denote the nonempty part.

An (m, n)-*Superhyperweighted Set* is a pair

$$(S, \mathcal{W}_{m,n}),$$

where the *weight assignment* is a map

$$\mathcal{W}_{m,n} :\ P_m(S) \longrightarrow P_{n+1}^*(\mathbb{R}).$$

Thus each m-level object $X \in P_m(S)$ (possibly a nested subset when $m \geq 1$) is assigned a nonempty $(n{+}1)$-level nested collection of real weights.

Remark 5.8.7. When $m = 0$, the domain $P_0(S) = S$ is the original set of atoms, and the map reduces to $\mathcal{W}_{0,n} : S \to P_{n+1}^*(\mathbb{R})$, i.e. a weight system on elements of S with $n{+}1$ nested weight layers.

Example 5.8.8 (A $(1, 1)$-Superhyperweighted set on subsets of S)**.** Let $S = \{a, b, c\}$ and take $(m, n) = (1, 1)$. Then $P_m(S) = P_1(S) = \mathcal{P}(S)$ and the codomain is $P_{n+1}^*(\mathbb{R}) = P_2^*(\mathbb{R}) = \mathcal{P}(\mathcal{P}(\mathbb{R})) \setminus \{\emptyset\}$.

Define $\mathcal{W}_{1,1} : \mathcal{P}(S) \to P_2^*(\mathbb{R})$ by

$$\mathcal{W}_{1,1}(A) := \Big\{\ \{\,|A|\,\},\ [\,|A| - 0.2,\ |A| + 0.2\,]\ \Big\} \qquad (A \subseteq S),$$

where $|A|$ denotes the cardinality of A. For example,

$$\mathcal{W}_{1,1}(\{a, b\}) = \big\{\{2\}, [1.8, 2.2]\big\}, \qquad \mathcal{W}_{1,1}(\emptyset) = \big\{\{0\}, [-0.2, 0.2]\big\}.$$

Each value is a nonempty family of nonempty subsets of $\mathbb{R}$, hence lies in $P_2^*(\mathbb{R})$. Therefore $(S, \mathcal{W}_{1,1})$ is a concrete $(1, 1)$-Superhyperweighted Set: it assigns nested weight families to *subsets* of S.

Theorem 5.8.9 (The $\boldsymbol{n}$-Superhyperweighted Set is a special case)**.** *Let S be a nonempty set and let $n \in \mathbb{N}_0$. Every n-Superhyperweighted Set $(S, \mathcal{W}_n)$ in the sense of Definition* (given above) *is precisely a special case of an (m, n)-Superhyperweighted Set with $m = 0$.*

Proof. By the given definition, an n-Superhyperweighted Set consists of a map

$$\mathcal{W}_n : \ S \longrightarrow P_{n+1}(\mathbb{R}) \setminus \{\emptyset\} = P^*_{n+1}(\mathbb{R}).$$

On the other hand, by Definition 5.8.6 with $m = 0$, an $(0, n)$-Superhyperweighted Set is a pair $(S, \mathcal{W}_{0,n})$ where

$$\mathcal{W}_{0,n} : \ P_0(S) \longrightarrow P^*_{n+1}(\mathbb{R}).$$

Since $P_0(S) = S$ by definition of iterated powersets, the types of $\mathcal{W}_n$ and $\mathcal{W}_{0,n}$ coincide exactly. Taking $\mathcal{W}_{0,n} := \mathcal{W}_n$ yields an $(0, n)$-Superhyperweighted Set, and conversely any $(0, n)$-Superhyperweighted Set restricts to such a map on S. Therefore the n-Superhyperweighted Set is exactly the $m = 0$ instance of the (m, n) framework. □

5.9 SuperHyper Z-Number

A Z-Number represents uncertain information using two fuzzy components: A (value constraint) and R (reliability) [334, 335]. Due to its conceptual simplicity and versatility, Z-Numbers have been extensively studied in various research areas [334, 336]. The definition is provided below. For further details on the definition of fuzzy numbers, readers are encouraged to consult [337,338] and related references.

Definition 5.9.1 (Z-Number)**.** [335] A *Z-number* is an ordered pair of fuzzy numbers, denoted as $Z = (A, R)$. This construct models uncertain information by capturing both a value's restriction and its associated reliability.

- A: The first component A is a fuzzy number representing a restriction on the possible values of a real-valued uncertain variable X. It defines the range of possible values that X can take under given conditions.

- R: The second component R is a fuzzy number representing the reliability or confidence level of the restriction A. It does not measure probability but instead reflects the certainty or sureness about the validity of the values indicated by A.

In summary, a Z-number (A, R) expresses that the value X is constrained by A with a confidence level given by R.

The Hyper Z-Number, introduced in [282, 339], is an extension of the Z-Number utilizing the framework of hyperstructures. Its definition is provided below.

Definition 5.9.2 (Hyper Z-Number)**.** [282] Let $\mathbb{R}$ denote the set of real numbers, and let $\tilde{P}(\mathbb{F})$ represent the family of all non-empty subsets of the set of fuzzy numbers $\mathbb{F}$. A *Hyper Z-Number* is a mapping:

$$\tilde{Z} : \mathbb{R} \to \tilde{P}(\mathbb{F} \times \mathbb{F}),$$

such that for each $x \in \mathbb{R}$, $\tilde{Z}(x)$ is a set of ordered pairs (A, B), where:

- A: A fuzzy number that provides a restriction on the possible values of x, representing the constraint on x's range or magnitude.

- B: A fuzzy number that specifies the reliability, confidence, or sureness of the restriction A. This serves as a secondary constraint indicating the reliability of the primary restriction.

A Hyper Z-Number generalizes the Z-number by allowing multiple pairs (A, B) for a given x, enabling the representation of multi-faceted or layered uncertainty.

Example 5.9.3 (Hyper Z-Number for temperature forecasts)**.** Let $\mathbb{F}$ denote the set of fuzzy numbers on $\mathbb{R}$. Consider the real variable x representing "tomorrow's maximum temperature" (in °C).

For each $x \in \mathbb{R}$, define $\tilde{Z}(x) \subseteq \mathbb{F} \times \mathbb{F}$ as a small set of alternative (A, R)-pairs, where A is a fuzzy constraint for the plausible temperature and R is a fuzzy reliability.

For example, at the abstract query point $x =$ "tomorrow", set

$$\tilde{Z}(x) = \Big\{ (A_{\text{mild}}, R_{\text{high}}),\ (A_{\text{warm}}, R_{\text{med}}) \Big\},$$

where A_{mild} is a triangular fuzzy number centered near 18 (say $(16, 18, 20)$), A_{warm} is a triangular fuzzy number centered near 24 (say $(22, 24, 26)$), and $R_{\text{high}}, R_{\text{med}}$ are fuzzy reliabilities (e.g. triangular on $[0, 1]$ centered near 0.9 and 0.6, respectively). Thus $\tilde{Z}(x)$ contains multiple admissible (A, R)-pairs, encoding layered uncertainty. Therefore $\tilde{Z} : \mathbb{R} \to \tilde{P}(\mathbb{F} \times \mathbb{F})$ is a concrete Hyper Z-Number.

The n-SuperHyper Z-Number is an extended concept derived from the Hyper Z-Number. Its definition is presented below [339].

Definition 5.9.4 (n-SuperHyper Z-Number)**.** [339] Let $\mathbb{R}$ denote the set of real numbers, and let $\mathcal{P}_n(\mathbb{F})$ represent the n-th powerset of the set of fuzzy numbers $\mathbb{F}$. An *n-SuperHyper Z-Number* is a mapping:

$$\tilde{Z}_n : \mathbb{R} \to \mathcal{P}_n(\mathbb{F} \times \mathbb{F}),$$

where for each $x \in \mathbb{R}$, $\tilde{Z}_n(x)$ is an element of the n-th powerset $\mathcal{P}_n(\mathbb{F} \times \mathbb{F})$. Specifically:

$$\tilde{Z}_n(x) = \{Z_1, Z_2, \ldots, Z_k\} \quad \text{where} \quad Z_i \in \mathcal{P}_{n-1}(\mathbb{F} \times \mathbb{F}),$$

and each element Z_i is a set of ordered pairs (A, R), with:

- A: A fuzzy number representing a constraint on the possible values of x. It reflects the range of possible values x can take under certain conditions.

- R: A fuzzy number representing the reliability, confidence, or sureness of the constraint A. It quantifies the certainty associated with A.

Base Case ($n = 1$): For $n = 1$, the n-SuperHyper Z-Number reduces to a Hyper Z-Number:

$$\tilde{Z}_1(x) \in \mathcal{P}(\mathbb{F} \times \mathbb{F}),$$

where:

$$\tilde{Z}_1(x) = \{(A_1, R_1), (A_2, R_2), \ldots, (A_m, R_m)\}.$$

Inductive Case ($n > 1$): For $n > 1$, $\tilde{Z}_n(x)$ is a set of nested structures, each element being a member of the $(n-1)$-th powerset:

$$\tilde{Z}_n(x) = \{\{(A_1, R_1), (A_2, R_2)\}, \{(A_3, R_3), (A_4, R_4)\}, \ldots\}.$$

Example 5.9.5 (A 2-SuperHyper Z-Number)**.** Let $\mathbb{F}$ be the set of fuzzy numbers on $\mathbb{R}$ and fix $n = 2$. Then $\mathcal{P}_2(\mathbb{F} \times \mathbb{F}) = \mathcal{P}(\mathcal{P}(\mathbb{F} \times \mathbb{F}))$, so $\tilde{Z}_2(x)$ is a *set of sets* of (A, R)-pairs.

For the same temperature variable $x =$ "tomorrow", define

$$\tilde{Z}_2(x) = \Big\{ Z^{(1)}(x),\ Z^{(2)}(x) \Big\},$$

where

$$Z^{(1)}(x) = \Big\{ \big(A_{\text{mild}}, R_{\text{high}}\big),\ \big(A_{\text{warm}}, R_{\text{med}}\big) \Big\}, \qquad Z^{(2)}(x) = \Big\{ \big(A_{\text{cool}}, R_{\text{high}}\big) \Big\}.$$

Here A_{cool} may be triangular around 14 (e.g. $(12, 14, 16)$). Thus $\tilde{Z}_2(x)$ expresses a second-level uncertainty: one scenario contains two alternative (A, R)-pairs, while another scenario contains a single highly reliable cool forecast. Hence $\tilde{Z}_2 : \mathbb{R} \to \mathcal{P}_2(\mathbb{F} \times \mathbb{F})$ is a concrete 2-SuperHyper Z-Number.

Definition 5.9.6 ((m, n)-SuperHyper Z-Number)**.** Let $\mathbb{F}$ denote the set of (real) fuzzy numbers, and let $m, n \in \mathbb{N}_0$. Using the iterated powersets $\mathcal{P}^0(X) = X$ and $\mathcal{P}^{t+1}(X) = \mathcal{P}(\mathcal{P}^t(X))$, an (m, n)-*SuperHyper Z-Number* is a mapping

$$\widetilde{Z}_{m,n} :\ \mathcal{P}^m(\mathbb{R}) \longrightarrow \mathcal{P}^n(\mathbb{F} \times \mathbb{F}),$$

such that, for every input $X \in \mathcal{P}^m(\mathbb{R})$, each *leaf-level* element of $\widetilde{Z}_{m,n}(X)$ (after flattening the outer n powerset layers) is an ordered pair $(A, R) \in \mathbb{F} \times \mathbb{F}$, where:

- $A \in \mathbb{F}$ is a fuzzy-number restriction on the values represented by X;
- $R \in \mathbb{F}$ is a fuzzy-number reliability (confidence/sureness) associated with A.

(If one prefers a classical-type convention, replace $\mathcal{P}^n$ by the nonempty iterated powerset $\mathcal{P}^n_*$ and require $\widetilde{Z}_{m,n}(X) \neq \emptyset$ for all X.)

Example 5.9.7 (A $(1,2)$-SuperHyper Z-Number)**.** Fix $m = 1$ and $n = 2$. Then the domain is $\mathcal{P}^1(\mathbb{R}) = \mathcal{P}(\mathbb{R})$, so the input is a set $X \subseteq \mathbb{R}$ of candidate real values (e.g. a coarse measurement band), and the codomain is $\mathcal{P}^2(\mathbb{F} \times \mathbb{F})$, a set of sets of (A, R)-pairs.

Let $X := [15, 25] \subseteq \mathbb{R}$ represent an admissible temperature range (in °C). Define

$$\widetilde{Z}_{1,2}(X) = \Big\{ \Big\{ \big(A_{\text{range}}, R_{\text{high}}\big),\ \big(A_{\text{range}}, R_{\text{med}}\big) \Big\},\ \Big\{ \big(A_{\text{peak}}, R_{\text{med}}\big) \Big\} \Big\}.$$

Here A_{range} is a fuzzy restriction supported on $[15, 25]$ (e.g. trapezoidal), A_{peak} is a fuzzy restriction peaked near 20, and $R_{\text{high}}, R_{\text{med}}$ are fuzzy reliabilities as in Example 5.9.3. Thus $\widetilde{Z}_{1,2}(X)$ is a 2-level nested family whose leaf elements are (A, R)-pairs, so it is a concrete $(1, 2)$-SuperHyper Z-Number.

Theorem 5.9.8 (The n-SuperHyper Z-Number is a special case)**.** *Let $n \in \mathbb{N}$. Every n-SuperHyper Z-Number*

$$\widetilde{Z}_n : \ \mathbb{R} \longrightarrow \mathcal{P}^n(\mathbb{F} \times \mathbb{F})$$

is precisely an (m, n)-SuperHyper Z-Number with $m = 0$. Equivalently, the class of n-SuperHyper Z-Numbers coincides with the class of $(0, n)$-SuperHyper Z-Numbers.

Proof. By definition of iterated powersets, $\mathcal{P}^0(\mathbb{R}) = \mathbb{R}$. Hence a map $\widetilde{Z}_n : \mathbb{R} \to \mathcal{P}^n(\mathbb{F} \times \mathbb{F})$ has exactly the same type as a map

$$\widetilde{Z}_{0,n} : \ \mathcal{P}^0(\mathbb{R}) \longrightarrow \mathcal{P}^n(\mathbb{F} \times \mathbb{F})$$

in Definition 5.9.6. Taking $\widetilde{Z}_{0,n} := \widetilde{Z}_n$ yields an $(0, n)$-SuperHyper Z-Number. Conversely, any $(0, n)$-SuperHyper Z-Number is, upon identifying $\mathcal{P}^0(\mathbb{R})$ with $\mathbb{R}$, an n-SuperHyper Z-Number. Therefore the n-SuperHyper Z-Number is exactly the $m = 0$ instance of the (m, n)-framework. □

5.10 SuperHyper D-Number

D-Number is a mapping from 2^Ω to $[0, 1]$ that assigns belief masses to nonempty subsets possibly summing below one, modeling incomplete evidence [340–347]. A Hyper D-number assigns each subset of Ω a set of masses in $[0, 1]$, with bounded sup-sum. A SuperHyper D-number maps nested collections of subsets to nested mass-sets, enforcing nested-null preservation and bounded deepest-level sup-sum.

Definition 5.10.1 (D-number)**.** [340–342] Let $\Omega = \{F_1, F_2, \ldots, F_N\}$ be a finite nonempty set, referred to as the *frame of discernment*, whose elements F_i are **not necessarily mutually exclusive**. A *D-number* on Ω is a mapping

$$d : 2^\Omega \to [0, 1]$$

satisfying the following two conditions:

(i) **Null-set constraint:** The empty set is assigned no belief mass:

$$d(\emptyset) = 0.$$

(ii) **Total mass constraint:** The sum of $d(A)$ over all nonempty subsets $A \subseteq \Omega$ is bounded above by 1:

$$\sum_{\substack{A \subseteq \Omega \\ A \neq \emptyset}} d(A) \leq 1.$$

Thus, a D-number allows the allocation of belief mass to any nonempty subset of Ω, where the total mass can be *strictly less than 1*, reflecting incomplete information. This generalizes the Dempster–Shafer basic probability assignment (BPA), which requires mutual exclusivity of elements and exact mass sum of 1.

Definition 5.10.2 (Hyper D-number)**.** Let $\Omega = \{F_1, \ldots, F_N\}$ be a finite nonempty frame of discernment. Denote by

$$P([0,1]) \;=\; \{\, A \mid A \subseteq [0,1]\}$$

the full powerset of the unit interval. A *Hyper D-number* on Ω is a mapping

$$D_H : 2^{\Omega} \;\longrightarrow\; P([0,1]), \qquad A \;\mapsto\; D_H(A) \subseteq [0,1],$$

subject to the following generalized constraints:

(i) **Null-set constraint:**

$$D_H(\emptyset) = \emptyset.$$

(ii) **Total-mass constraint:**

$$\bigcup_{\substack{A \subseteq \Omega \\ A \neq \emptyset}} D_H(A) \;\subseteq\; [0,1], \quad \sum_{\substack{A \subseteq \Omega \\ A \neq \emptyset}} \sup D_H(A) \;\leq 1.$$

Here $\sup D_H(A)$ denotes the least upper bound of the set $D_H(A) \subseteq [0,1]$.

Example 5.10.3 (A Hyper D-number on a two-element frame)**.** Let $\Omega = \{F_1, F_2\}$. Then $2^{\Omega} = \{\emptyset, \{F_1\}, \{F_2\}, \{F_1, F_2\}\}$. Define $D_H : 2^{\Omega} \to \mathcal{P}([0,1])$ by

$$D_H(\emptyset) = \emptyset, \qquad D_H(\{F_1\}) = [0.10, 0.20],$$

$$D_H(\{F_2\}) = \{0.30, 0.35\}, \qquad D_H(\{F_1, F_2\}) = [0.05, 0.10].$$

Each value is a subset of $[0,1]$, and the null-set constraint holds by construction.

For the total-mass constraint, compute the suprema:

$$\sup D_H(\{F_1\}) = 0.20, \quad \sup D_H(\{F_2\}) = 0.35, \quad \sup D_H(\{F_1, F_2\}) = 0.10.$$

Hence

$$\sum_{\substack{A \subseteq \Omega \\ A \neq \emptyset}} \sup D_H(A) = 0.20 + 0.35 + 0.10 = 0.65 \;\leq\; 1.$$

Therefore D_H is a concrete Hyper D-number on Ω.

Definition 5.10.4 (SuperHyper D-number)**.** Let $\Omega = \{F_1, \dots, F_N\}$ be a finite nonempty frame of discernment. For each integer $n \geq 0$, write

$$\mathcal{P}^n\big(2^\Omega\big) = \underbrace{\mathcal{P}\big(\mathcal{P}(\cdots \mathcal{P}(2^\Omega)\cdots)\big)}_{n \text{ iterations}}, \quad \mathcal{P}^n\big([0,1]\big) = \underbrace{\mathcal{P}\big(\mathcal{P}(\cdots \mathcal{P}([0,1])\cdots)\big)}_{n \text{ iterations}}.$$

A *SuperHyper D-number of order* n is a mapping

$$D_{SH} : \mathcal{P}^n\big(2^\Omega\big) \longrightarrow \mathcal{P}^n\big([0,1]\big),$$

subject to the generalized constraints:

(i) **Null-set constraint:** $D_{SH}\big(\underbrace{\dots\{\emptyset\}\dots}_{n \text{ times}}\big) = \underbrace{\dots\{\emptyset\}\dots}_{n \text{ times}}$.

(ii) **Total-mass constraint:**

$$\sum_{\substack{B \in \mathcal{P}^n(2^\Omega) \\ B \neq \emptyset}} \sup\big(D_{SH}(B)\big) \leq 1,$$

where sup extracts the least upper bound at the deepest level.

Example 5.10.5 (A 2-SuperHyper D-number)**.** Let $\Omega = \{F_1, F_2\}$ and fix $n = 2$. Then $\mathcal{P}^2(2^\Omega) = \mathcal{P}(\mathcal{P}(2^\Omega))$ and $\mathcal{P}^2([0,1]) = \mathcal{P}(\mathcal{P}([0,1]))$.

Define $D_{SH} : \mathcal{P}^2(2^\Omega) \to \mathcal{P}^2([0,1])$ as follows. First, set the canonical nested empty input and output:

$$\varnothing_2 = \{\{\emptyset\}\} \in \mathcal{P}^2(2^\Omega), \qquad D_{SH}(\varnothing_2) = \varnothing_2 \in \mathcal{P}^2([0,1]).$$

Next, define D_{SH} on two representative nonempty inputs:

$$B_1 := \big\{\{F_1\}, \{F_2\}\big\} \in \mathcal{P}(2^\Omega) \subseteq \mathcal{P}^2(2^\Omega), \qquad D_{SH}(B_1) := \big\{\{0.10, 0.15\}, [0.12, 0.18]\big\},$$

$$B_2 := \big\{\{F_1, F_2\}\big\} \in \mathcal{P}^2(2^\Omega), \qquad D_{SH}(B_2) := \big\{\{0.05\}\big\}.$$

(Any other values may be assigned similarly.)

The deepest-level suprema are

$$\sup\big(D_{SH}(B_1)\big) = \max\{0.15, 0.18\} = 0.18, \qquad \sup\big(D_{SH}(B_2)\big) = 0.05.$$

Thus the total-mass constraint holds on the displayed support:

$$\sup(D_{SH}(B_1)) + \sup(D_{SH}(B_2)) = 0.18 + 0.05 = 0.23 \leq 1,$$

and D_{SH} is a concrete 2-SuperHyper D-number in the sense of the definition.

Definition 5.10.6 ((m,n)-SuperHyper D-Number)**.** Let $\Omega = \{F_1, \dots, F_N\}$ be a finite nonempty frame of discernment and write $2^\Omega = \mathcal{P}(\Omega)$. Fix integers $m, n \in \mathbb{N}_0$. Let $\mathcal{P}^0(X) = X$ and $\mathcal{P}^{t+1}(X) = \mathcal{P}(\mathcal{P}^t(X))$ be iterated powersets.

An (m,n)-*SuperHyper D-Number* on Ω is a mapping

$$D_{m,n} : \ \mathcal{P}^m(2^\Omega) \longrightarrow \mathcal{P}^n([0,1])$$

satisfying the following constraints:

(i) **Null-object constraint.** Let $\varnothing_m \in \mathcal{P}^m(2^\Omega)$ denote the canonical "empty" m-level object, defined recursively by

$$\varnothing_0 := \emptyset, \qquad \varnothing_{t+1} := \{\varnothing_t\} \quad (t \geq 0).$$

Then

$$D_{m,n}(\varnothing_m) = \varnothing_n, \qquad \text{where } \varnothing_n := \emptyset \text{ if } n = 0, \text{ and } \varnothing_{t+1} := \{\varnothing_t\}.$$

(ii) **Total-mass constraint (upper-envelope form).** Define the "supremum-at-leaves" functional $\sup_n : \mathcal{P}^n([0,1]) \to [0,1]$ recursively by

$$\sup_0(x) := x \quad (x \in [0,1]), \qquad \sup_{t+1}(A) := \sup\{\sup_t(a) \mid a \in A\} \quad (A \in \mathcal{P}^{t+1}([0,1])),$$

with the convention $\sup_{t+1}(\emptyset) = 0$. Then the total mass of $D_{m,n}$ is bounded by

$$\sum_{\substack{B \in \mathcal{P}^m(2^\Omega) \\ B \neq \varnothing_m}} \sup_n\big(D_{m,n}(B)\big) \;\leq\; 1.$$

If one prefers a classical-type convention, one may additionally require $D_{m,n}(B) \neq \varnothing$ for all $B \neq \varnothing_m$.

Example 5.10.7 (A $(1,2)$-SuperHyper D-number)**.** Let $\Omega = \{F_1, F_2\}$ and take $(m,n) = (1,2)$. Then the domain is $\mathcal{P}^1(2^\Omega) = \mathcal{P}(2^\Omega)$ (families of subsets of Ω), and the codomain is $\mathcal{P}^2([0,1])$ (sets of subsets of $[0,1]$).

Define $D_{1,2} : \mathcal{P}(2^\Omega) \to \mathcal{P}^2([0,1])$ by

$$D_{1,2}(\emptyset) := \emptyset,$$

and for two representative nonempty inputs

$$U_1 := \big\{\{F_1\}, \{F_1, F_2\}\big\}, \qquad D_{1,2}(U_1) := \big\{\{0.20\}, [0.15, 0.25]\big\},$$

$$U_2 := \big\{\{F_2\}\big\}, \qquad D_{1,2}(U_2) := \big\{\{0.30, 0.35\}\big\}.$$

Then

$$\sup_2\big(D_{1,2}(U_1)\big) = 0.25, \qquad \sup_2\big(D_{1,2}(U_2)\big) = 0.35,$$

so the total-mass bound holds on this support:

$$\sup_2(D_{1,2}(U_1)) + \sup_2(D_{1,2}(U_2)) = 0.25 + 0.35 = 0.60 \leq 1.$$

Therefore $D_{1,2}$ is a concrete $(1,2)$-SuperHyper D-number on Ω.

Theorem 5.10.8 (The n-SuperHyper D-Number is a special case)**.** *Fix $n \in \mathbb{N}_0$. Every n-SuperHyper D-Number*

$$D_{SH} : \; \mathcal{P}^n(2^\Omega) \longrightarrow \mathcal{P}^n([0,1])$$

satisfying the null-object and total-mass constraints in the given definition of SuperHyper D-number is an (m,n)-SuperHyper D-Number with $m = n$. In particular, the class of n-SuperHyper D-Numbers embeds into the class of (m,n)-SuperHyper D-Numbers.

Proof. Take $m := n$ and define $D_{m,n} := D_{SH}$. Then $D_{m,n}$ has the required type

$$D_{m,n} : \mathcal{P}^m(2^\Omega) = \mathcal{P}^n(2^\Omega) \longrightarrow \mathcal{P}^n([0,1]).$$

The null-object constraints coincide because both definitions use the canonical nested empty object $\varnothing_n = \{\cdots\{\emptyset\}\cdots\}$ (nested n times) and require that it map to itself.

For the total-mass constraint, the definition of (m,n)-SuperHyper D-Number uses $\sup_n\big(D_{m,n}(B)\big)$, which is exactly the "extract the least upper bound at the deepest level" rule used in the SuperHyper D-number definition (written there informally as taking sup at the deepest layer). Hence the inequality

$$\sum_{B \neq \varnothing_n} \sup_n\big(D_{SH}(B)\big) \leq 1$$

is identical to the required constraint for $D_{m,n}$. Therefore D_{SH} is an (n,n)-SuperHyper D-Number. □

5.11 SuperHyperGrey Set

A *Grey set* on a nonempty universe X is a function $G : X \to [0,1]$, where $G(x)$ denotes the greyness degree assigned to $x \in X$ [348–351]. A *HyperGrey set* on X is a mapping $G^{\mathrm{H}} : X \to \mathcal{P}([0,1])$, so that each element $x \in X$ is assigned a (possibly non-singleton) set $G^{\mathrm{H}}(x) \subseteq [0,1]$ of admissible greyness degrees [352]. A *SuperHyperGrey set of order n* on X is a function $G^{\mathrm{SH}} : \mathcal{P}^n(X) \to \mathcal{P}^n([0,1])$, assigning to each $A \in \mathcal{P}^n(X)$ an n-nested collection $G^{\mathrm{SH}}(A) \in \mathcal{P}^n([0,1])$ of greyness degrees [352].

Definition 5.11.1 (Grey Set)**.** [348, 349] Let X be a nonempty universe of discourse. A *Grey set* on X is a function

$$G : X \to [0,1],$$

where for each $x \in X$, the value $G(x) \in [0,1]$ represents the *greyness degree* of x (e.g., $G(x) = 0$ means completely white; $G(x) = 1$ means completely black; and $0 < G(x) < 1$ means intermediate greyness).

Definition 5.11.2 (HyperGrey Set)**.** Let X be a nonempty universe of discourse. A *HyperGrey set* on X is a mapping

$$G^{\mathrm{H}} : X \to \mathcal{P}([0,1]),$$

where for each $x \in X$, the subset $G^{\mathrm{H}}(x) \subseteq [0,1]$ represents the set of possible greyness degrees of x.

Definition 5.11.3 (SuperHyperGrey Set of order n)**.** Let X be a nonempty universe and fix an integer $n \geq 0$. Define the iterated powersets by

$$\mathcal{P}^0(X) := X, \qquad \mathcal{P}^{k+1}(X) := \mathcal{P}(\mathcal{P}^k(X)) \quad (k \geq 0).$$

A *SuperHyperGrey set of order n* on X is a function

$$G^{\mathrm{SH}} : \mathcal{P}^n(X) \to \mathcal{P}^n([0,1]),$$

assigning to each $A \in \mathcal{P}^n(X)$ an n-nested collection $G^{\mathrm{SH}}(A) \in \mathcal{P}^n([0,1])$ of greyness degrees.

Example 5.11.4 (A concrete SuperHyperGrey set of order 2)**.** Let

$$X = \{x_1, x_2, x_3\}$$

be a nonempty universe. Take $n = 2$, so $\mathcal{P}^2(X) = \mathcal{P}(\mathcal{P}(X))$ and $\mathcal{P}^2([0,1]) = \mathcal{P}(\mathcal{P}([0,1]))$.

Define $G^{\mathrm{SH}} : \mathcal{P}^2(X) \to \mathcal{P}^2([0,1])$ as follows. For each $A \in \mathcal{P}^2(X)$, interpret A as a *family of groups of objects* (a set of subsets of X), and assign to it a *family of sets of admissible greyness degrees* (a set of subsets of $[0,1]$).

For instance, let

$$A_1 := \big\{\{x_1, x_2\}, \{x_3\}\big\} \in \mathcal{P}^2(X).$$

Set

$$G^{\mathrm{SH}}(A_1) := \big\{\, [0.10, 0.30],\ [0.60, 0.80] \,\big\} \in \mathcal{P}^2([0,1]),$$

meaning that the first group $\{x_1, x_2\}$ is assessed to have low-to-moderate greyness, while the second group $\{x_3\}$ is assessed to have high greyness.

Similarly, for

$$A_2 := \big\{\{x_2\}, \{x_1, x_3\}\big\} \in \mathcal{P}^2(X),$$

define

$$G^{\mathrm{SH}}(A_2) := \big\{\, [0.20, 0.40],\ [0.45, 0.55] \,\big\} \in \mathcal{P}^2([0,1]).$$

Since each output is a set of subsets of $[0,1]$, we indeed have $G^{\mathrm{SH}}(A_i) \in \mathcal{P}^2([0,1])$ for $i = 1, 2$. Hence G^{SH} is a SuperHyperGrey set of order 2 on X in the sense of Definition 5.11.3.

Chapter 6

HyperStructure and SuperHyperStructure in Other Mathematics

Beyond the areas discussed so far, a wide variety of HyperStructures and SuperHyperStructures have also been studied. In this chapter, we describe HyperStructure and SuperHyperStructure in other branches of mathematics. For reference, Table 6.1 summarizes classical concepts and their hyper/superhyper counterparts in analysis, probability, and general structures.

6.1 HyperFunction and SuperHyperFunction

A hyperfunction maps each element $x \in S$ to a subset $f(x) \subseteq S$, allowing multiple outputs per input [220, 353]. A superhyperfunction maps m-level nested subsets $\mathcal{P}^m(S)$ to n-level nested subsets $\mathcal{P}^n(S)$, encoding hierarchical multivalued behavior [219, 354–357].

Definition 6.1.1 (Hyperfunction)**.** [220, 353] A *hyperfunction* on a set S is any function

$$f : S \longrightarrow \mathcal{P}(S).$$

For each $x \in S$, one has $f(x) \subseteq S$. Thus a single input may correspond to multiple outputs, unlike the classical single–valued case.

Example 6.1.2 (A hyperfunction on a three-element set)**.** Let $S = \{a, b, c\}$. Define a hyperfunction $f : S \to \mathcal{P}(S)$ by

$$f(a) = \{a, b\}, \qquad f(b) = \{b\}, \qquad f(c) = \{a, c\}.$$

Thus a single input may return multiple outputs; for example, $f(a)$ allows either a or b as admissible outputs. Hence f is a hyperfunction in the sense of the definition.

Definition 6.1.3 ((m, n)–Superhyperfunction)**.** (cf. [220, 357]) Let S be a nonempty set and $m, n \geq 1$. An (m, n)–*superhyperfunction* is a mapping

$$f : \mathcal{P}^m(S) \longrightarrow \mathcal{P}^n(S).$$

When $m = r \leq n$, this specializes to the usual notion of an n–superhyperfunction, namely a map $\mathcal{P}^r(S) \to \mathcal{P}^n(S)$.

Table 6.1: Classical concepts and their hyper/superhyper counterparts in analysis, probability, and general structures.

Classical Concept	**HyperConcept**	**SuperHyperConcept**
Function	HyperFunction	SuperHyperFunction
Probability	HyperProbability	SuperHyperProbability
Entropy	HyperEntropy	SuperHyperEntropy
Metastructure	Meta-Hyperstructure	Meta-SuperHyperStructure
Ordered Structure	Ordered HyperStructure	Ordered SuperHyperStructure
Integral	HyperIntegral	SuperHyperIntegral
Space	HyperSpace	SuperHyperSpace
Interval	HyperInterval	SuperHyperInterval
Curried Structure	Curried HyperStructure	Curried SuperHyperStructure

Example 6.1.4 (A $(1,2)$-superhyperfunction induced by union and complement)**.** Let $S = \{a, b\}$ and take $(m, n) = (1, 2)$. Then

$$\mathcal{P}^1(S) = \mathcal{P}(S), \qquad \mathcal{P}^2(S) = \mathcal{P}(\mathcal{P}(S)).$$

Define $F : \mathcal{P}(S) \to \mathcal{P}(\mathcal{P}(S))$ by

$$F(A) := \{\, A,\ S \setminus A \,\} \qquad (A \subseteq S).$$

Each value $F(A)$ is a set whose elements are subsets of S, hence $F(A) \in \mathcal{P}^2(S)$. For example,

$$F(\{a\}) = \big\{\{a\}, \{b\}\big\}, \qquad F(\emptyset) = \big\{\emptyset, \{a, b\}\big\}.$$

Therefore F is a concrete $(1,2)$-superhyperfunction on S.

6.2 HyperProbability and SuperHyperProbability

Standard probability theory assigns each event a single real value in $[0, 1]$, satisfying non-negativity, normalization, and countable additivity (cf. [358–363]). Related concepts such as fuzzy probability [364,365] and neutrosophic probability [366–368] are also known. HyperProbability assigns a set of probability values to each event, capturing uncertainty from multiple sources [238, 369]. The definitions and examples of HyperProbability are presented below.

Definition 6.2.1 (Probability Space)**.** (cf. [370–372]) A *probability space* is a triple $(\Omega, \mathcal{F}, P)$ where

- Ω is a nonempty set called the *sample space*;
- $\mathcal{F}$ is a σ-algebra of subsets of Ω, called the *event space*;
- $P : \mathcal{F} \to [0, 1]$ is a function, called a *probability measure*, satisfying:
  1. $P(A) \geq 0$ for all $A \in \mathcal{F}$ (non-negativity),

2. $P(\Omega) = 1$ (normalization),

3. For any countable sequence of pairwise disjoint events $A_1, A_2, \cdots \in \mathcal{F}$,

$$P\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \sum_{i=1}^{\infty} P(A_i) \quad \text{(countable additivity)}.$$

Definition 6.2.2 (Hyper-Probability)**.** [238, 369] Let Ω be a sample space and $\mathcal{F}$ a σ-algebra of events. A *Hyper-Probability* is a function

$$\mathrm{HP} : \mathcal{F} \longrightarrow \mathcal{P}([0,1]),$$

where $\mathcal{P}([0,1])$ denotes the power set of the unit interval. For each event $A \in \mathcal{F}$,

$$\mathrm{HP}(A) \;=\; \{\, p_k(A) \mid k \in K_A \},$$

with each classical probability $p_k(A) \in [0,1]$. Thus $\mathrm{HP}(A)$ collects multiple probability assessments for the same event.

Example 6.2.3 (Hyper-Probability on two coin-toss events)**.** Let $\Omega = \{H, T\}$ be the outcome space of a single coin toss and let $\mathcal{F} = \mathcal{P}(\Omega)$. Consider two classical probability assessments:

$$p_1(H) = 0.5 \quad \text{(fair coin)}, \qquad p_2(H) = 0.6 \quad \text{(biased coin)}.$$

Define the Hyper-Probability $\mathrm{HP} : \mathcal{F} \to \mathcal{P}([0,1])$ by

$$\mathrm{HP}(A) := \{p_1(A), p_2(A)\} \qquad (A \in \mathcal{F}),$$

where $p_i(A) = \sum_{\omega \in A} p_i(\{\omega\})$. Then, for instance,

$$\mathrm{HP}(\{H\}) = \{0.5, 0.6\}, \qquad \mathrm{HP}(\{T\}) = \{0.5, 0.4\}, \qquad \mathrm{HP}(\Omega) = \{1, 1\}.$$

Thus HP collects multiple probability values for the same event, and is a concrete Hyper-Probability in the sense of Definition 6.2.2.

n-SuperHyperProbability generalizes HyperProbability by iteratively applying powersets, modeling layered uncertainty across multiple decision or belief levels [238]. The definitions and examples of n-SuperHyperProbability are presented below.

Definition 6.2.4 (n-SuperHyperProbability)**.** [238] For $n \geq 1$, an *n-SuperHyperProbability* is a function

$$\mathrm{SHP}^{(n)} : \mathcal{F} \longrightarrow P^n([0,1]),$$

where $P^n([0,1])$ denotes the n-fold iterated power set of $[0,1]$. It is defined recursively by

$$\mathrm{SHP}^{(1)}(A) = \mathrm{HP}(A), \qquad \mathrm{SHP}^{(n)}(A) = P\big(\mathrm{SHP}^{(n-1)}(A)\big) \quad (n \geq 2).$$

In particular, $\mathrm{SHP}^{(1)}$ coincides with Hyper-Probability and for $n = 0$ one recovers the classical single-valued probability.

Example 6.2.5 (A 2-SuperHyperProbability)**.** Let $(\Omega, \mathcal{F})$ and HP be as in Example 6.2.3. Fix $n = 2$. By Definition 6.2.4,

$$\mathrm{SHP}^{(2)}(A) = \mathcal{P}\big(\mathrm{SHP}^{(1)}(A)\big) = \mathcal{P}\big(\mathrm{HP}(A)\big) \qquad (A \in \mathcal{F}).$$

For the event $A = \{H\}$, we have $\mathrm{HP}(\{H\}) = \{0.5, 0.6\}$, hence

$$\mathrm{SHP}^{(2)}(\{H\}) = \mathcal{P}(\{0.5, 0.6\}) = \big\{\emptyset, \{0.5\}, \{0.6\}, \{0.5, 0.6\}\big\} \in \mathcal{P}^2([0,1]).$$

Thus $\mathrm{SHP}^{(2)}$ returns a *set of possible probability-sets* for each event, i.e. a nested uncertainty representation.

A (m,n)-SuperHyperProbability is a set-valued mapping sending m-fold event collections to n-fold iterated probability sets, thereby modeling hierarchical and multi-source uncertainty across multiple distinct levels systematically.

Definition 6.2.6 ((m,n)-SuperHyperProbability)**.** Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $\mathrm{HP} : \mathcal{F} \to \mathcal{P}([0,1])$ be a Hyper-Probability measure as in Definition 6.2.2. For integers $m, n \geq 0$, define recursively a map

$$\mathrm{SHP}^{(m,n)} : \underbrace{\mathcal{P}\big(\mathcal{P}(\cdots \mathcal{P}}_{m \text{ times}}(\mathcal{F}) \cdots)\big) \longrightarrow \underbrace{\mathcal{P}\big(\mathcal{P}(\cdots \mathcal{P}}_{n \text{ times}}([0,1]) \cdots)\big)$$

as follows:

(1) Base on the "vertical'' index.

$$\mathrm{SHP}^{(0,0)}(A) = P(A), \qquad \mathrm{SHP}^{(0,1)}(A) = \mathrm{HP}(A),$$

and for $n \geq 2$,

$$\mathrm{SHP}^{(0,n)}(A) = \mathcal{P}\big(\mathrm{SHP}^{(0,n-1)}(A)\big) \subseteq \mathcal{P}^n([0,1]), \quad A \in \mathcal{F}.$$

Thus $\mathrm{SHP}^{(0,n)}$ coincides with the usual n-SuperHyperProbability.

(2) Recursion on the "horizontal'' index. For $m \geq 1$, any $U \in \mathcal{P}^m(\mathcal{F})$ is a (finite) set of elements of $\mathcal{P}^{m-1}(\mathcal{F})$. We set

$$\mathrm{SHP}^{(m,n)}(U) = \bigcup_{A \in U} \mathrm{SHP}^{(m-1,n)}(A).$$

Since each $\mathrm{SHP}^{(m-1,n)}(A)$ lies in $\mathcal{P}^n([0,1])$, their union is again an element of $\mathcal{P}^n([0,1])$.

Example 6.2.7 (A $(1,2)$-SuperHyperProbability aggregating event-families)**.** Let $(\Omega, \mathcal{F})$ and the Hyper-Probability HP be as in Example 6.2.3, and take $(m,n) = (1,2)$. Then the domain is $\mathcal{P}^1(\mathcal{F}) = \mathcal{P}(\mathcal{F})$, i.e. families of events, and the codomain is $\mathcal{P}^2([0,1])$.

Consider the event-family

$$U := \big\{\{H\}, \{T\}\big\} \in \mathcal{P}(\mathcal{F}).$$

By the recursion rule in Definition 6.2.6 (union aggregation),

$$\mathrm{SHP}^{(1,2)}(U) = \bigcup_{A \in U} \mathrm{SHP}^{(0,2)}(A) = \mathrm{SHP}^{(0,2)}(\{H\}) \ \cup \ \mathrm{SHP}^{(0,2)}(\{T\}).$$

Now $\mathrm{HP}(\{H\}) = \{0.5, 0.6\}$ and $\mathrm{HP}(\{T\}) = \{0.5, 0.4\}$, so

$$\mathrm{SHP}^{(0,2)}(\{H\}) = \mathcal{P}(\{0.5, 0.6\}), \qquad \mathrm{SHP}^{(0,2)}(\{T\}) = \mathcal{P}(\{0.5, 0.4\}).$$

Hence $\mathrm{SHP}^{(1,2)}(U)$ is the union of these two families of probability-sets, i.e. a well-defined element of $\mathcal{P}^2([0,1])$. Interpretation: the output aggregates second-order probability descriptions across a *collection* of events.

6.3 HyperEntropy and SuperHyperEntropy

Entropy measures uncertainty or information content of a probability distribution, quantified as expected surprise, typically using Shannon's formula $-\sum p \log p$ [373, 374]. Hyperentropy is Shannon entropy of a probability mass function on subsets of S, quantifying uncertainty over set-valued outcomes [375]. n-Superhyperentropy is Shannon entropy of a probability mass function on $\mathcal{P}^n(S)$, measuring uncertainty over nested subset outcomes [375]. The definition of HyperEntropy is given as follows.

Definition 6.3.1 (Hyperentropy)**.** [375] Let S be a finite set and $\mathcal{P}(S)$ its powerset. Suppose

$$p \colon \mathcal{P}(S) \ \longrightarrow \ [0,1]$$

is a probability mass function on subsets of S, i.e. $\sum_{A \subseteq S} p(A) = 1$. The *hyperentropy* of p is defined by

$$H_{\mathrm{h}}(p) \ = \ -\sum_{A \subseteq S} p(A) \, \log p(A),$$

where log denotes the natural logarithm (nats) or base-2 logarithm (bits).

Example 6.3.2 (Hyperentropy on a small powerset)**.** Let $S = \{a, b\}$. Then

$$\mathcal{P}(S) = \big\{\emptyset, \{a\}, \{b\}, \{a, b\}\big\}.$$

Define a probability mass function $p : \mathcal{P}(S) \to [0, 1]$ by

$$p(\emptyset) = \tfrac{1}{2}, \qquad p(\{a\}) = \tfrac{1}{4}, \qquad p(\{b\}) = \tfrac{1}{8}, \qquad p(\{a, b\}) = \tfrac{1}{8},$$

so that $\sum_{A\subseteq S} p(A) = 1$. Using base-2 logarithms (so entropy is measured in bits), the hyperentropy is

$$H_{\mathrm{h}}(p) = -\sum_{A\subseteq S} p(A)\log_2 p(A) = -\Big(\tfrac{1}{2}\log_2\tfrac{1}{2} + \tfrac{1}{4}\log_2\tfrac{1}{4} + \tfrac{1}{8}\log_2\tfrac{1}{8} + \tfrac{1}{8}\log_2\tfrac{1}{8}\Big).$$

Since $\log_2(1/2) = -1$, $\log_2(1/4) = -2$, and $\log_2(1/8) = -3$, we obtain

$$H_{\mathrm{h}}(p) = \tfrac{1}{2}\cdot 1 + \tfrac{1}{4}\cdot 2 + \tfrac{1}{8}\cdot 3 + \tfrac{1}{8}\cdot 3 = \tfrac{7}{4} = 1.75 \text{ bits}.$$

The definition of SuperHyperEntropy is given as follows.

Definition 6.3.3 (n-Superhyperentropy)**.** [375] Let $n \in \mathbb{N}$ and S a finite set. Write

$$V^{(n)} = \mathcal{P}^n(S), \quad M_n = \big|V^{(n)}\big|.$$

Let $p\colon V^{(n)} \to [0, 1]$ be a probability mass function on $V^{(n)}$, so that $\sum_{A\in V^{(n)}} p(A) = 1$. The *$n$-superhyperentropy* $H_n(p)$ is

$$H_n(p) \;=\; -\sum_{A\in V^{(n)}} p(A)\,\log p(A).$$

Example 6.3.4 (2-superhyperentropy on $\mathcal{P}^2(S)$)**.** Let $S = \{a\}$ and fix $n = 2$. Then

$$V^{(2)} = \mathcal{P}^2(S) = \mathcal{P}(\mathcal{P}(S)) = \Big\{\emptyset,\ \{\emptyset\},\ \{\{a\}\},\ \{\emptyset, \{a\}\}\Big\}.$$

Define the uniform distribution $p : V^{(2)} \to [0, 1]$ by $p(A) = \frac{1}{4}$ for all $A \in V^{(2)}$. Using base-2 logarithms, the 2-superhyperentropy is

$$H_2(p) = -\sum_{A\in V^{(2)}} p(A)\log_2 p(A) = -4\cdot\tfrac{1}{4}\log_2\tfrac{1}{4} = -\log_2\tfrac{1}{4} = 2 \text{ bits}.$$

6.4 Metastructures, Meta-Hyperstructures, and Meta-Superhyperstructures

Metastructures formalize the idea of a "structure whose elements are structures." The essential requirement is that the allowed *meta-operations* combine whole objects (structures) by *uniform rules* and are *isomorphism-invariant* (i.e., they depend only on the input objects up to isomorphism) [376–378].

Fix a single-sorted finitary signature

$$\Sigma = \big(\mathsf{Func}, \mathsf{Rel}, \mathrm{ar}_{\mathsf{Func}}, \mathrm{ar}_{\mathsf{Rel}}\big),$$

and write Str_Σ for the class of Σ-structures.

Definition 6.4.1 (Metastructure)**.** [378] A *metastructure* over Σ is a pair

$$\mathbb{M} = \big(U, (\Phi_\ell)_{\ell\in\Lambda}\big),$$

where $U \subseteq \mathrm{Str}_\Sigma$ is nonempty and each meta-operation $\Phi_\ell : U^{k_\ell} \to U$ satisfies:

- **(Closure)** $\Phi_\ell(\vec{\mathbf{C}}) \in U$ for all $\vec{\mathbf{C}} \in U^{k_\ell}$;
- **(Uniformity)** the output carrier and symbol-interpretations are built from the inputs by a fixed constructor (same recipe for all inputs);
- **(Iso-invariance)** if $\mathbf{C}_i \cong \mathbf{D}_i$ for all i, then $\Phi_\ell(\mathbf{C}_1, \ldots, \mathbf{C}_{k_\ell}) \cong \Phi_\ell(\mathbf{D}_1, \ldots, \mathbf{D}_{k_\ell})$.

Definition 6.4.2 (Meta-hyperstructure over a signature Σ)**.** Let Σ be a fixed (single-sorted, finitary) signature, and let HypStr_Σ denote the class of Σ-hyperstructures (up to isomorphism). Write $\mathcal{P}_*(U) := \mathcal{P}(U) \setminus \{\emptyset\}$ for the nonempty powerset.

A *meta-hyperstructure* over Σ is a pair

$$\mathbb{MH} = \big(U, (\Phi_\ell)_{\ell\in\Lambda}\big),$$

where $U \subseteq \mathrm{HypStr}_\Sigma$ is nonempty and, for each label $\ell \in \Lambda$ with meta-arity $k_\ell \in \mathbb{N}$, the map

$$\Phi_\ell : U^{k_\ell} \longrightarrow \mathcal{P}_*(U)$$

is a *meta-hyperoperation* satisfying the following axioms:

- **(MH1) Closure.** For every $(\mathbf{H}_1, \ldots, \mathbf{H}_{k_\ell}) \in U^{k_\ell}$ we have $\Phi_\ell(\mathbf{H}_1, \ldots, \mathbf{H}_{k_\ell}) \subseteq U$.
- **(MH2) Uniform construction.** There exist fixed (input-independent) constructors

  $$\Gamma_\ell, \quad \Lambda_\ell^f, \quad \Xi_\ell^R, \quad \Pi_\ell^q$$

  (for carriers, function symbols, relation symbols, and hyperoperations, respectively) such that for every output $\mathbf{K} \in \Phi_\ell(\mathbf{H}_1, \ldots, \mathbf{H}_{k_\ell})$, the carrier and all interpretations in $\mathbf{K}$ are obtained from those of $\mathbf{H}_1, \ldots, \mathbf{H}_{k_\ell}$ by applying these constructors.
- **(MH3) Isomorphism-invariance (naturality).** If $\alpha_i : \mathbf{H}_i \cong \mathbf{H}_i'$ for $1 \le i \le k_\ell$, then there exists an induced bijection

  $$\Phi_\ell(\alpha_1, \ldots, \alpha_{k_\ell}) : \Phi_\ell(\mathbf{H}_1, \ldots, \mathbf{H}_{k_\ell}) \xrightarrow{\cong} \Phi_\ell(\mathbf{H}_1', \ldots, \mathbf{H}_{k_\ell}')$$

  whose values are hyperstructure isomorphisms (preserving all symbols of Σ and all designated hyperoperations).

Definition 6.4.3 (Meta-superhyperstructure of profile Θ over a base set H)**.** Fix a nonempty base set H and a profile $\Theta = (m, n, k)$ with $m, n \in \mathbb{N}_0$ and $k \in \mathbb{N}$. Let $\mathrm{SH}_\Theta(H)$ denote the class of Θ-superhyperstructures over H (up to isomorphism induced by bijections of H), i.e., objects $\mathbf{S} = (\mathcal{P}^m(H), \star_{\mathbf{S}})$ with $\star_{\mathbf{S}} : (\mathcal{P}^m(H))^k \to \mathcal{P}^n(H)$.

A *meta-superhyperstructure* of profile Θ over H is a pair

$$\mathbb{MSH} = \big(U, (\Phi_\ell)_{\ell \in \Lambda}\big),$$

where $U \subseteq \mathrm{SH}_\Theta(H)$ is nonempty and, for each label $\ell \in \Lambda$, there are fixed parameters

$$s_\ell \in \mathbb{N} \quad \text{(meta-arity)}, \qquad r_\ell \in \mathbb{N}_0 \quad \text{(meta-height)},$$

together with a map

$$\Phi_\ell : U^{s_\ell} \longrightarrow \mathcal{P}^{r_\ell}(U)$$

satisfying:

- **(MSH1) Closure.** For every $(\mathbf{S}_1, \dots, \mathbf{S}_{s_\ell}) \in U^{s_\ell}$, every element of $\Phi_\ell(\mathbf{S}_1, \dots, \mathbf{S}_{s_\ell})$ lies in $\mathcal{P}^{r_\ell}(U)$.

- **(MSH2) Uniform constructor on internal operations.** There exists a fixed constructor
$$\Pi_\ell : \Big[\ (\mathcal{P}^m(H))^k \to \mathcal{P}^n(H)\ \Big]^{s_\ell} \longrightarrow \Big[\ (\mathcal{P}^m(H))^k \to \mathcal{P}^n(H)\ \Big]$$
such that whenever $\mathbf{T}$ is an output object occurring (at the appropriate nesting level) in $\Phi_\ell(\mathbf{S}_1, \dots, \mathbf{S}_{s_\ell})$, its internal superhyperoperation satisfies
$$\star_{\mathbf{T}} = \Pi_\ell\big(\star_{\mathbf{S}_1}, \dots, \star_{\mathbf{S}_{s_\ell}}\big).$$

- **(MSH3) Naturality under base bijections.** For any bijection $\alpha : H \to H$, let α act on $\mathcal{P}^m(H)$ and $\mathcal{P}^n(H)$ by image, and let $\mathbf{S} \mapsto \mathbf{S}^\alpha$ denote transport of Θ-superhyperstructures along α. Then transporting inputs along α transports outputs along α: every output object $\mathbf{T}$ occurring in $\Phi_\ell(\mathbf{S}_1, \dots, \mathbf{S}_{s_\ell})$ corresponds to an output object $\mathbf{T}^\alpha$ occurring in $\Phi_\ell(\mathbf{S}_1^\alpha, \dots, \mathbf{S}_{s_\ell}^\alpha)$, and
$$\star_{\mathbf{T}^\alpha} = \alpha^{[n]} \circ \star_{\mathbf{T}} \circ \big(\alpha^{[m]}\big)^{-1},$$
where $\alpha^{[m]}$ and $\alpha^{[n]}$ are the induced maps on $\mathcal{P}^m(H)$ and $\mathcal{P}^n(H)$.

Example 6.4.4 (Real-world example: composing multi-level access-control policies as a meta-superhyperstructure)**.** Let H be the set of *atomic permissions* in a cloud service, e.g.,

$$H = \{\mathsf{read{:}db}, \mathsf{write{:}db}, \mathsf{read{:}logs}, \mathsf{deploy}, \mathsf{admin}, \dots\}.$$

A *role policy* (e.g. Engineer, SRE, Auditor) is not just a flat subset of H: in practice it is built from *groups of permissions* (bundles) and sometimes from bundles-of-bundles (e.g. "incident-response" contains "read-logs bundle" and "deploy bundle").

Fix the profile $\Theta = (m, n, k) = (1, 2, 2)$. Then $\mathcal{P}^m(H) = \mathcal{P}(H)$ represents permission bundles, and $\mathcal{P}^n(H) = \mathcal{P}^2(H) = \mathcal{P}(\mathcal{P}(H))$ represents *collections of bundles* (i.e. a policy described as a set of permission-bundles). A Θ-superhyperstructure over H is an object

$$\mathbf{S} = \big(\mathcal{P}(H), \star_{\mathbf{S}}\big), \qquad \star_{\mathbf{S}} : \mathcal{P}(H) \times \mathcal{P}(H) \to \mathcal{P}^2(H),$$

where $\star_{\mathbf{S}}(A, B)$ returns a *set of candidate policy-bundles* obtained by combining two bundles $A, B \subseteq H$.

Two concrete internal superhyperoperations. Consider two teams with different security philosophies:

- *Conservative merge* (least privilege): $\mathbf{S}_1 = (\mathcal{P}(H), \star_1)$ where

$$\star_1(A, B) := \{\, A \cap B \,\},$$

  so combining two bundles yields the single bundle of permissions allowed by *both*.

- *Operational merge* (availability-first): $\mathbf{S}_2 = (\mathcal{P}(H), \star_2)$ where

$$\star_2(A, B) := \{\, A \cup B,\ A \cup \mathrm{BreakGlass} \,\},$$

  where $\mathrm{BreakGlass} \subseteq H$ is a fixed emergency bundle (e.g. temporary admin with audit), so the output is a *set* of possible policy bundles to choose from depending on incident severity.

Meta-level combination (meta-superhyperoperation). Let $U \subseteq \mathrm{SH}_\Theta(H)$ contain admissible policy-combiners used by different departments. Define a meta-superhyperoperation $\Phi : U^2 \to \mathcal{P}(U)$ (meta-height $r = 1$) by returning a set of *hybrid* combiners. Concretely, for $\mathbf{S}_1, \mathbf{S}_2 \in U$ let

$$\Phi(\mathbf{S}_1, \mathbf{S}_2) := \{\mathbf{T}\}, \quad \text{where} \quad \mathbf{T} = \big(\mathcal{P}(H), \star_{\mathbf{T}}\big) \text{ and } \star_{\mathbf{T}} = \Pi(\star_1, \star_2),$$

and the uniform constructor Π is defined pointwise by

$$\Pi(\star_1, \star_2)(A, B) := \star_1(A, B) \ \cup \ \star_2(A, B) \qquad (A, B \in \mathcal{P}(H)).$$

Thus $\mathbf{T}$ enumerates policy options produced either by the conservative rule or the operational rule, and the meta-level object $\Phi(\mathbf{S}_1, \mathbf{S}_2)$ represents the organizational choice of *how to combine* policy-combiners.

Why this matches Definition 6.4.3. (MSH1) holds because Φ outputs an element of U (hence a member of $\mathcal{P}(U)$). (MSH2) holds because the constructor Π is fixed and input-independent. For (MSH3), if permissions are renamed by a bijection $\alpha : H \to H$ (e.g. refactoring API scopes), then bundles and bundle-collections are transported by image, and the hybrid constructor remains equivariant:

$$\alpha^{[2]}\big(\Pi(\star_1, \star_2)(A, B)\big) = \Pi(\star_1^{\alpha}, \star_2^{\alpha})\big(\alpha^{[1]}(A), \alpha^{[1]}(B)\big),$$

so the meta-superhyperstructure is natural under renaming of atomic permissions.

6.5 Ordered HyperStructure and Ordered SuperHyperStructure

An Ordered HyperStructure is a hyperstructure equipped with a partial order on its base set, where the hyperoperations are required to be monotone with respect to this order, ensuring that enlarging inputs in the order never decreases outputs (cf. [379–381]).

Definition 6.5.1 (Lifted order on subsets)**.** Let $(S, \leq)$ be a poset. For nonempty $U, V \subseteq S$ define

$$U \preccurlyeq V \quad :\Longleftrightarrow \quad \forall u \in U\ \exists v \in V \text{ such that } u \leq v.$$

(When $U = \emptyset$, we stipulate $\emptyset \preccurlyeq V$ for all V.)

Definition 6.5.2 (Ordered hyperstructure)**.** [379, 380] An *ordered hyperstructure* is a triple $(S, \circ, \leq)$ such that

(i) $(S, \leq)$ is a partially ordered set;

(ii) $\circ$ is a (binary) hyperoperation on S;

(iii) (*monotonicity*) whenever $x \leq y$ in S and $z \in S$, one has

$$z \circ x \ \preccurlyeq\ z \circ y \quad \text{and} \quad x \circ z \ \preccurlyeq\ y \circ z,$$

where $\preccurlyeq$ is the lifted order from above.

Example 6.5.3 (An ordered hyperstructure on $(\mathbb{N}, \leq)$)**.** Let $S = \mathbb{N}$ with the usual order $\leq$. Define a (total) binary hyperoperation $\circ : S \times S \to \mathcal{P}_*(S)$ by

$$x \circ y \ := \ \{\, t \in \mathbb{N} \mid t \geq x + y \,\} \qquad (x, y \in \mathbb{N}).$$

Thus $x \circ y$ is the upward-closed set (principal upset) generated by $x + y$, hence nonempty.

Let $\preccurlyeq$ be the Hoare (forward) lifting of $\leq$ to subsets of S, i.e. $A \preccurlyeq B$ iff $\forall a \in A\, \exists b \in B$ with $a \leq b$.

We verify monotonicity. If $x \leq y$ and $z \in \mathbb{N}$, then $z + x \leq z + y$, hence

$$z \circ x = \{t \mid t \geq z + x\} \ \preccurlyeq\ \{t \mid t \geq z + y\} = z \circ y,$$

because for any $t \geq z + x$ we may choose $t' := \max\{t, z + y\} \in z \circ y$ with $t \leq t'$. Similarly,

$$x \circ z = \{t \mid t \geq x + z\} \ \preccurlyeq\ \{t \mid t \geq y + z\} = y \circ z.$$

Therefore $(S, \circ, \leq)$ is an ordered hyperstructure in the sense of Definition 6.5.2.

An Ordered SuperHyperStructure extends Ordered hyperStructure to superhyperstructures over iterated powersets, enriching them with a partial order and demanding that the superhyperoperations be coordinatewise monotone, thereby strictly generalizing ordered hyperstructures.

Definition 6.5.4 (Hoare (forward) lifting of the order)**.** For $k \geq 0$ define a binary relation $\preccurlyeq_k$ on $\mathcal{P}^k(S)$ by recursion:

$$\text{for } k = 0: \quad x \preccurlyeq_0 y \iff x \leq y; \qquad x, y \in S,$$
$$\text{for } k \to k+1: \quad X \preccurlyeq_{k+1} Y \iff \big(\forall\, x \in X\ \exists\, y \in Y:\ x \preccurlyeq_k y\big), \qquad X, Y \in \mathcal{P}^{k+1}(S).$$

(We stipulate $\emptyset \preccurlyeq_{k+1} Y$ for all Y.)

Definition 6.5.5 (Ordered SuperHyperStructure of order (m,n))**.** An *Ordered SuperHyperStructure of order* (m,n) *and arity* s is a triple

$$\big(S,\ \odot^{(m,n)},\ \leq\big)$$

such that:

(i) $(S,\leq)$ is a poset;

(ii) $\odot^{(m,n)} : (\mathcal{P}^m(S))^s \to \mathcal{P}^n(S)$ is given;

(iii) (*coordinatewise monotonicity*) for each coordinate $j \in \{1,\dots,s\}$ and all inputs

$$X_1,\dots,X_s,\ Y_j \in \mathcal{P}^m(S),$$

if $X_j \preccurlyeq_m Y_j$ and $X_i = Y_i$ for $i \neq j$, then

$$\odot^{(m,n)}(X_1,\dots,X_j,\dots,X_s)\ \preccurlyeq_n\ \odot^{(m,n)}(X_1,\dots,Y_j,\dots,X_s).$$

Equivalently, $\odot^{(m,n)}$ is (jointly) monotone with respect to the product preorder $\preccurlyeq_m^{\times s}$ on the domain and $\preccurlyeq_n$ on the codomain.

Example 6.5.6 (An ordered $(1,1)$-SuperHyperStructure induced by unions on $(\mathbb{N},\leq)$)**.** Let $S=\mathbb{N}$ with the usual order $\leq$. Take $m=n=1$ and arity $s=2$. Thus the domain is $(\mathcal{P}(S))^2$ and the codomain is $\mathcal{P}(S)$. Use the Hoare lifting $\preccurlyeq_1$ on $\mathcal{P}(S)$:

$$A \preccurlyeq_1 B \iff (\forall a\in A)\,(\exists b\in B:\ a\leq b).$$

Define a binary superhyperoperation $\odot^{(1,1)} : \mathcal{P}(S)\times\mathcal{P}(S)\to\mathcal{P}(S)$ by

$$X \odot^{(1,1)} Y\ :=\ X\cup Y.$$

(So $\odot^{(1,1)}$ is simply union on the powerset level.)

We verify coordinatewise monotonicity. Suppose $X \preccurlyeq_1 X'$ and fix $Y\subseteq S$. Take any $u\in X\cup Y$. If $u\in Y$, then $u\in X'\cup Y$ and $u\leq u$ witnesses the Hoare condition. If $u\in X$, then by $X\preccurlyeq_1 X'$ there exists $u'\in X'$ with $u\leq u'$; since $u'\in X'\cup Y$, the Hoare condition is again satisfied. Hence

$$X\odot^{(1,1)}Y = X\cup Y\ \preccurlyeq_1\ X'\cup Y = X'\odot^{(1,1)}Y.$$

The argument with the other coordinate is symmetric, so $\odot^{(1,1)}$ is coordinatewise monotone.

Therefore $\big(S,\odot^{(1,1)},\leq\big)$ is an ordered SuperHyperStructure of order $(1,1)$ and arity 2 in the sense of Definition 6.5.5.

6.6 HyperIntegral and SuperHyperIntegral

HyperIntegral aggregates a set-valued function over a finite hyperfield using hyperaddition, producing one nonempty subset as result [382]. SuperHyperIntegral sums a function $P_m(H) \to P_n(H)$ via level-n lifted hyperaddition, yielding an n-nested output [382].

Definition 6.6.1 (HyperIntegral in a hyperfield)**.** Let $F = (H, \oplus, \otimes, 0, 1)$ be a (commutative) hyperfield and assume H is finite. Let $f : H \to P^*(H)$ be a set-valued function (so $f(x) \neq \varnothing$ for all $x \in H$).

Define the *hyper-sum* of two nonempty subsets $A, B \subseteq H$ by

$$A \boxplus B \;:=\; \bigcup_{a\in A}\; \bigcup_{b\in B} (a \oplus b) \;\in P^*(H),$$

and define the iterated hyper-sum of a finite family $(A_i)_{i=1}^{t} \subseteq P^*(H)$ recursively by

$$\boxplus_{i=1}^{1} A_i := A_1, \qquad \boxplus_{i=1}^{t} A_i := \Big(\boxplus_{i=1}^{t-1} A_i\Big) \boxplus A_t \quad (t \geq 2).$$

The *HyperIntegral* of f over H is the element of $P^*(H)$ defined by

$$\int_H f(x) \;\oplus\; dx \;:=\; \boxplus_{x\in H} f(x).$$

Example 6.6.2 (HyperIntegral on the Krasner hyperfield)**.** Let $H = \{0, 1\}$ be the Krasner hyperfield, with hyperaddition $\oplus$ given by

$$0 \oplus 0 = \{0\}, \qquad 0 \oplus 1 = 1 \oplus 0 = \{1\}, \qquad 1 \oplus 1 = \{0, 1\}.$$

Define a set-valued function $f : H \to P^*(H)$ by

$$f(0) = \{0\}, \qquad f(1) = \{1\}.$$

Compute the HyperIntegral

$$\int_H f(x) \;\oplus\; dx = \boxplus_{x\in H} f(x) = f(0) \boxplus f(1).$$

By the definition of $\boxplus$,

$$f(0) \boxplus f(1) = \bigcup_{a\in\{0\}}\; \bigcup_{b\in\{1\}} (a \oplus b) = 0 \oplus 1 = \{1\}.$$

Hence the HyperIntegral equals $\{1\} \in P^*(H)$.

Definition 6.6.3 (Level-n lifted hyperaddition and hyper-sum)**.** Let $F = (H, \oplus, \otimes, 0, 1)$ be a hyperfield and fix $n \geq 1$. Assume the iterated powersets $P_n(H)$ have been defined already.

Define a binary operation $\boxplus_n : P_n(H) \times P_n(H) \to P_n(H)$ recursively as follows.

- For $n = 1$ and $A, B \in P_1(H) = P(H)$, set

$$A \boxplus_1 B \;:=\; \bigcup_{a\in A}\ \bigcup_{b\in B} (a \oplus b) \;\in P(H).$$

- For $n \geq 2$ and $\mathcal{A}, \mathcal{B} \in P_n(H) = P(P_{n-1}(H))$, set

$$\mathcal{A} \boxplus_n \mathcal{B} \;:=\; \Big\{ A \boxplus_{n-1} B \;\Big|\; A \in \mathcal{A},\ B \in \mathcal{B} \Big\} \;\in P_n(H).$$

For a finite family $(X_i)_{i=1}^{t} \subseteq P_n(H)$, define the iterated sum

$$\boxplus_{i=1}^{1,n} X_i := X_1, \qquad \boxplus_{i=1}^{t,n} X_i := \Big(\boxplus_{i=1}^{t-1,n} X_i \Big) \boxplus_n X_t \quad (t \geq 2).$$

Example 6.6.4 (Level-2 lifted hyper-sum on the Krasner hyperfield)**.** Let $H = \{0, 1\}$ be the Krasner hyperfield and fix $n = 2$. Then $P_2(H) = \mathcal{P}(\mathcal{P}(H))$, so elements of $P_2(H)$ are sets of subsets of H.

Take, for instance,

$$\mathcal{A} := \big\{\{0\}, \{1\}\big\} \in P_2(H), \qquad \mathcal{B} := \big\{\{1\}\big\} \in P_2(H).$$

By the recursive definition,

$$\mathcal{A} \boxplus_2 \mathcal{B} = \Big\{ A \boxplus_1 B \;\Big|\; A \in \mathcal{A},\ B \in \mathcal{B} \Big\}.$$

Now $\mathcal{B}$ has only $B = \{1\}$, so we compute:

$$\{0\} \boxplus_1 \{1\} = \bigcup_{a\in\{0\},\, b\in\{1\}} (a \oplus b) = 0 \oplus 1 = \{1\},$$

$$\{1\} \boxplus_1 \{1\} = \bigcup_{a\in\{1\},\, b\in\{1\}} (a \oplus b) = 1 \oplus 1 = \{0, 1\}.$$

Therefore

$$\mathcal{A} \boxplus_2 \mathcal{B} = \big\{\{1\}, \{0, 1\}\big\} \in P_2(H).$$

Definition 6.6.5 ((m, n)-SuperHyperIntegral)**.** Let $F = (H, \oplus, \otimes, 0, 1)$ be a hyperfield and assume H is finite. Fix integers $m, n \geq 1$. Let

$$f :\ P_m(H) \longrightarrow P_n(H)$$

be a function on the m-th powerset level with values in the n-th powerset level.

Using the level-n iterated hyper-sum $\boxplus^n$ from Definition 6.6.3, define the (m, n)-*SuperHyperIntegral* of f by

$$\int_{P_m(H)} f(A) \;\oplus\; dA \;:=\; \boxplus_{A\in P_m(H)}^{n} f(A) \;\in\; P_n(H).$$

If one wishes to exclude empty outputs, replace $P_n(H)$ by $P_n^*(H)$ and assume $f(A) \in P_n^*(H)$ for all A.

Example 6.6.6 (A $(1,2)$-SuperHyperIntegral)**.** Let $H = \{0,1\}$ be the Krasner hyperfield and take $(m,n) = (1,2)$. Then

$$P_1(H) = \mathcal{P}(H), \qquad P_2(H) = \mathcal{P}(\mathcal{P}(H)).$$

Define a function $f : P_1(H) \to P_2(H)$ by

$$f(A) := \{\, A,\ H \,\} \qquad (A \subseteq H).$$

Thus, for example,

$$f(\emptyset) = \{\emptyset, H\}, \quad f(\{0\}) = \big\{\{0\}, H\big\}, \quad f(\{1\}) = \big\{\{1\}, H\big\}, \quad f(H) = \{H\}.$$

The $(1,2)$-SuperHyperIntegral is

$$\int_{P_1(H)} f(A) \ \oplus\ dA = \overset{2}{\underset{A \in P_1(H)}{\boxplus}} f(A),$$

i.e. the level-2 iterated $\boxplus_2$-sum of the four values $f(\emptyset), f(\{0\}), f(\{1\}), f(H)$. For instance, summing the first two terms gives

$$f(\emptyset) \boxplus_2 f(\{0\}) = \Big\{\, X \boxplus_1 Y \ \Big|\ X \in f(\emptyset),\ Y \in f(\{0\}) \Big\} \in P_2(H),$$

where each $X, Y \in \{\emptyset, \{0\}, \{1\}, H\}$, and $X \boxplus_1 Y$ is computed using the Krasner hyperaddition. Continuing recursively (in any chosen order, since H is finite) produces a well-defined element of $P_2(H)$, which is the desired $(1,2)$-SuperHyperIntegral.

6.7 HyperSpace and SuperHyperSpace

HyperSpace is the hyperspace $(\mathcal{P}_*(X), \tau_H)$, whose points are nonempty subsets of X with a hypertopology [382]. SuperHyperSpace is $(\mathcal{P}_*^n(X), \tau_{SH}^{(n)})$, a topology on nested nonempty subsets, generalizing hyperspaces to level n [382].

Definition 6.7.1 (HyperSpace)**.** [382] Let $X \neq \emptyset$ be a set. A *HyperSpace* on X is a topological space

$$\mathrm{Hyp}(X) \ :=\ \big(\mathcal{P}_*(X),\ \tau_H\big),$$

where $\mathcal{P}_*(X)$ is the family of all nonempty subsets of X (as defined earlier), and τ_H is a topology on the underlying set $\mathcal{P}_*(X)$ (i.e., a *hypertopology* on X). Elements of $\mathcal{P}_*(X)$ are regarded as the "points" of the hyperspace.

Example 6.7.2 (A finite HyperSpace on a two-point set)**.** Let $X = \{a, b\}$. Then

$$\mathcal{P}_*(X) = \big\{\{a\}, \{b\}, \{a,b\}\big\}.$$

Define a topology τ_H on the underlying set $\mathcal{P}_*(X)$ by

$$\tau_H := \big\{\emptyset,\ \mathcal{P}_*(X),\ U_a,\ U_b,\ U_{ab}\big\},$$

where

$$U_a := \big\{A \in \mathcal{P}_*(X) \mid a \in A\big\} = \big\{\{a\}, \{a,b\}\big\},$$
$$U_b := \big\{A \in \mathcal{P}_*(X) \mid b \in A\big\} = \big\{\{b\}, \{a,b\}\big\},$$
$$U_{ab} := U_a \cup U_b = \big\{\{a\}, \{b\}, \{a,b\}\big\} = \mathcal{P}_*(X).$$

Then τ_H contains $\emptyset$ and $\mathcal{P}_*(X)$, is closed under arbitrary unions, and is closed under finite intersections (indeed $U_a \cap U_b = \{\{a,b\}\}$, which is open because $\{\{a,b\}\} = \mathcal{P}_*(X) \setminus (\{\{a\}\} \cup \{\{b\}\})$ in this finite setting). Hence $(\mathcal{P}_*(X), \tau_H)$ is a HyperSpace in the sense of Definition 6.7.1.

Definition 6.7.3 (n-SuperHyperSpace)**.** [382] Let $X \neq \emptyset$ be a set and fix an integer $n \geq 1$. An *n-SuperHyperSpace* on X is a topological space

$$\mathrm{SHyp}_n(X) \;:=\; \big(\mathcal{P}_*^n(X),\, \tau_{SH}^{(n)}\big),$$

where $\mathcal{P}_*^n(X)$ is the n-th iterated nonempty powerset (as defined earlier), and $\tau_{SH}^{(n)}$ is a topology on $\mathcal{P}_*^n(X)$ (i.e., a *superhypertopology of level n*). For $n = 1$, one recovers a HyperSpace: $\mathrm{SHyp}_1(X) = (\mathcal{P}_*(X), \tau_{SH}^{(1)})$.

Remark 6.7.4 (Optional canonical choice: Vietoris hyperspace)**.** If (X, τ) is a topological space, a standard HyperSpace is obtained by taking the carrier $\mathrm{CL}_*(X)$ of nonempty τ-closed subsets and equipping it with the Vietoris topology, generated by the subbasic sets

$$\langle U\rangle^+ := \{F \in \mathrm{CL}_*(X) \mid F \subseteq U\}, \qquad \langle U\rangle^- := \{F \in \mathrm{CL}_*(X) \mid F \cap U \neq \emptyset\}, \quad U \in \tau.$$

Example 6.7.5 (A level-2 SuperHyperSpace on $X = \{a, b\}$)**.** Let $X = \{a, b\}$ and fix $n = 2$. Then

$$\mathcal{P}_*^2(X) = \mathcal{P}_*\big(\mathcal{P}_*(X)\big) = \Big\{\; \mathcal{A} \subseteq \mathcal{P}_*(X) \;\Big|\; \mathcal{A} \neq \emptyset \;\Big\},$$

so points are nonempty *collections* of nonempty subsets of X.

Define a topology $\tau_{SH}^{(2)}$ on $\mathcal{P}_*^2(X)$ by the "membership subbasis": for each $B \in \mathcal{P}_*(X)$, let

$$\mathcal{U}_B := \big\{\mathcal{A} \in \mathcal{P}_*^2(X) \mid B \in \mathcal{A}\big\},$$

and let $\tau_{SH}^{(2)}$ be the topology generated by the subbasic family $\{\mathcal{U}_B : B \in \mathcal{P}_*(X)\}$. Explicitly, $\tau_{SH}^{(2)}$ consists of arbitrary unions of finite intersections of the sets $\mathcal{U}_B$.

For example,

$$\mathcal{U}_{\{a\}} \cap \mathcal{U}_{\{a,b\}} = \big\{\mathcal{A} \subseteq \mathcal{P}_*(X) \setminus \{\emptyset\} \mid \{a\}, \{a, b\} \in \mathcal{A}\big\}$$

is open in $\tau_{SH}^{(2)}$. Therefore $\big(\mathcal{P}_*^2(X), \tau_{SH}^{(2)}\big)$ is a level-2 SuperHyperSpace in the sense of Definition 6.7.3.

6.8 HyperInterval and SuperHyperInterval

An interval set is a set of real numbers containing every point between any two of its elements; convex subset [383–386]. Related concepts include interval-valued fuzzy sets [387–389] and interval-valued neutrosophic sets [390–393], among others. HyperInterval set maps each interval $[A_\ell, A_u]$ to a nonempty family of admissible interval sets on the same universe [394]. SuperHyperInterval set assigns to each n-nested subset $A \in \mathcal{P}^n(U)$ an $(n-1)$-nested family of interval sets.

Definition 6.8.1 (HyperInterval set)**.** Let U be a nonempty (finite) universe. Let $I(\mathcal{P}(U))$ denote the class of all interval sets on U, i.e., $[A_\ell, A_u] := \{A \subseteq U \mid A_\ell \subseteq A \subseteq A_u\}$ with $A_\ell \subseteq A_u \subseteq U$. A *HyperInterval set* on U is a map

$$\mathrm{HI} : I(\mathcal{P}(U)) \longrightarrow \mathcal{P}_*\big(I(\mathcal{P}(U))\big),$$

assigning to each base interval set $I = [A_\ell, A_u]$ a nonempty family $\mathrm{HI}(I) \subseteq I(\mathcal{P}(U))$ of admissible interval sets.

(*Optional refinement discipline.*) If, moreover, every $[B_\ell, B_u] \in \mathrm{HI}([A_\ell, A_u])$ satisfies

$$A_\ell \subseteq B_\ell \subseteq B_u \subseteq A_u,$$

then HI is called a *refinement HyperInterval set*.

Example 6.8.2 (A refinement HyperInterval set on a three-element universe)**.** Let $U = \{u_1, u_2, u_3\}$. Consider the base interval set

$$I = [A_\ell, A_u], \qquad A_\ell = \{u_1\}, \quad A_u = \{u_1, u_2, u_3\}.$$

Thus

$$[A_\ell, A_u] = \big\{A \subseteq U \mid \{u_1\} \subseteq A \subseteq U\big\} = \big\{\{u_1\}, \{u_1, u_2\}, \{u_1, u_3\}, U\big\}.$$

Define $\mathrm{HI} : I(\mathcal{P}(U)) \to \mathcal{P}_*(I(\mathcal{P}(U)))$ by specifying its value on $I = [A_\ell, A_u]$:

$$\mathrm{HI}\big([A_\ell, A_u]\big) := \Big\{\big[\{u_1\}, \{u_1, u_2\}\big],\ \big[\{u_1\}, \{u_1, u_3\}\big],\ \big[\{u_1, u_2\}, U\big]\Big\}.$$

Each element of $\mathrm{HI}([A_\ell, A_u])$ is itself an interval set on U, and the family is nonempty. Moreover, every $[B_\ell, B_u]$ listed satisfies

$$A_\ell = \{u_1\} \subseteq B_\ell \subseteq B_u \subseteq A_u = U,$$

so HI is a *refinement* HyperInterval set.

Definition 6.8.3 (SuperHyperInterval set of order n)**.** Let U be a nonempty (finite) universe and fix an integer $n \geq 1$. Using the iterated powersets $\mathcal{P}^0(U) = U$ and $\mathcal{P}^{t+1}(U) = \mathcal{P}(\mathcal{P}^t(U))$ for $t \geq 0$, a *SuperHyperInterval set of order n* on U is a map

$$\mathrm{SHI}^{(n)} : \ \mathcal{P}^n(U) \longrightarrow \mathcal{P}^{\,n-1}\big(I(\mathcal{P}(U))\big).$$

Thus, to each n-nested subset $A \in \mathcal{P}^n(U)$, the map assigns an $(n-1)$-nested family of interval sets on U.

Example 6.8.4 (A 2-SuperHyperInterval set)**.** Let $U = \{u_1, u_2\}$ and fix $n = 2$. Then

$$\mathcal{P}^2(U) = \mathcal{P}(\mathcal{P}(U)).$$

Choose a level-2 input

$$A := \big\{\{u_1\}, \{u_1, u_2\}\big\} \ \in \ \mathcal{P}^2(U).$$

A 2-SuperHyperInterval set is a map

$$\mathrm{SHI}^{(2)} : \mathcal{P}^2(U) \to \mathcal{P}^1\big(I(\mathcal{P}(U))\big) = I(\mathcal{P}(U)),$$

so it assigns to A an *ordinary* interval set on U.

Define $\mathrm{SHI}^{(2)}(A)$ to be the interval set

$$\mathrm{SHI}^{(2)}(A) := \big[\{u_1\}, \{u_1, u_2\}\big] = \big\{B \subseteq U \mid \{u_1\} \subseteq B \subseteq \{u_1, u_2\}\big\} = \big\{\{u_1\}, \{u_1, u_2\}\big\}.$$

Thus $\mathrm{SHI}^{(2)}$ maps a 2-nested subset A to an interval set on U, providing a concrete example of a SuperHyperInterval set of order 2.

6.9 Curried HyperStructures and SuperHyperstructures

Curried hyperstructure rewrites a k-ary hyperoperation $\circ$ as nested unary maps, preserving evaluation $\circ(x_1, \ldots, x_k)$ [395]. Curried superhyperstructure curries a k-ary (m, n)-superhyperoperation into iterated function spaces, keeping the same k-argument behavior [395]. These concepts apply the idea of currying (curried functions), originally used in functional programming languages (cf. [396–399]).

Definition 6.9.1 (Curried k-ary HyperStructure)**.** Let $S \neq \emptyset$ and let $\mathcal{P}(S)$ be its powerset (already defined). Fix an integer $k \geq 1$ and let

$$\circ : \big(\mathcal{P}(S)\big)^k \longrightarrow \mathcal{P}(S)$$

be a k-ary hyperoperation (i.e., the operation of a k-ary hyperstructure).

Define the *curried form* of $\circ$ to be the map

$$\operatorname{curry}(\circ) : \ \mathcal{P}(S) \longrightarrow \Big(\mathcal{P}(S) \to \big(\cdots \to (\mathcal{P}(S) \to \mathcal{P}(S)) \cdots\big)\Big),$$

characterized by the evaluation rule

$$\operatorname{curry}(\circ)(X_1)(X_2)\cdots(X_k) \ = \ \circ(X_1, X_2, \ldots, X_k) \qquad (X_1, \ldots, X_k \in \mathcal{P}(S)).$$

Then

$$\big(\mathcal{P}(S),\, \operatorname{curry}(\circ)\big)$$

is called the *curried k-ary HyperStructure* associated with $(\mathcal{P}(S), \circ)$.

Example 6.9.2 (A curried ternary HyperStructure on $\mathcal{P}(S)$)**.** Let $S = \{a, b\}$ and take $k = 3$. Define a ternary hyperoperation $\circ : (\mathcal{P}(S))^3 \to \mathcal{P}(S)$ by

$$\circ(X, Y, Z) := (X \cup Y) \cap Z \qquad (X, Y, Z \subseteq S).$$

This is well-defined since $X \cup Y \subseteq S$ and hence $(X \cup Y) \cap Z \subseteq S$.

Its curried form

$$\operatorname{curry}(\circ) : \mathcal{P}(S) \to \big(\mathcal{P}(S) \to (\mathcal{P}(S) \to \mathcal{P}(S))\big)$$

is given explicitly by

$$\operatorname{curry}(\circ)(X)(Y)(Z) = \circ(X, Y, Z) = (X \cup Y) \cap Z.$$

For example, with $X = \{a\}$, $Y = \{b\}$, $Z = \{a\}$,

$$\operatorname{curry}(\circ)(\{a\})(\{b\})(\{a\}) = (\{a\} \cup \{b\}) \cap \{a\} = \{a\}.$$

Thus $\big(\mathcal{P}(S), \operatorname{curry}(\circ)\big)$ is a concrete curried 3-ary HyperStructure.

Definition 6.9.3 (Curried k-ary (m,n)-SuperHyperStructure)**.** Let $S \neq \emptyset$, fix integers $m, n \geq 0$ and $k \geq 1$, and let $\mathcal{P}^m(S)$ and $\mathcal{P}^n(S)$ denote the iterated powersets (already defined). Let

$$\star : \big(\mathcal{P}^m(S)\big)^k \longrightarrow \mathcal{P}^n(S)$$

be a k-ary (m,n)-superhyperoperation (i.e., a SuperHyperStructure of order (m,n) and arity k).

Define the *curried form* of $\star$ by

$$\operatorname{curry}(\star) : \ \mathcal{P}^m(S) \longrightarrow \Big(\mathcal{P}^m(S) \to \big(\cdots \to (\mathcal{P}^m(S) \to \mathcal{P}^n(S)) \cdots\big)\Big),$$

with evaluation

$$\operatorname{curry}(\star)(X_1)(X_2)\cdots(X_k) \ = \ \star(X_1, X_2, \ldots, X_k) \qquad (X_1, \ldots, X_k \in \mathcal{P}^m(S)).$$

Then

$$\big(\mathcal{P}^m(S),\, \operatorname{curry}(\star)\big)$$

is called the *curried k-ary (m,n)-SuperHyperStructure* associated with $\star$.

Example 6.9.4 (A curried binary $(1,2)$-SuperHyperStructure)**.** Let $S = \{a, b\}$ and take $(m, n, k) = (1, 2, 2)$. Then $\mathcal{P}^m(S) = \mathcal{P}(S)$ and $\mathcal{P}^n(S) = \mathcal{P}^2(S) = \mathcal{P}(\mathcal{P}(S))$.

Define a binary $(1,2)$-superhyperoperation $\star : (\mathcal{P}(S))^2 \to \mathcal{P}^2(S)$ by

$$\star(X, Y) := \big\{\, X,\ Y,\ X \cup Y \,\big\} \qquad (X, Y \subseteq S).$$

Each element X, Y, and $X \cup Y$ is a subset of S, hence $\star(X,Y) \subseteq \mathcal{P}(S)$, so $\star(X,Y) \in \mathcal{P}(\mathcal{P}(S)) = \mathcal{P}^2(S)$.

The curried form

$$\operatorname{curry}(\star) : \mathcal{P}(S) \to \big(\mathcal{P}(S) \to \mathcal{P}^2(S)\big)$$

satisfies

$$\operatorname{curry}(\star)(X)(Y) = \star(X, Y) = \{X, Y, X \cup Y\}.$$

For example, with $X = \{a\}$ and $Y = \{b\}$,

$$\operatorname{curry}(\star)(\{a\})(\{b\}) = \big\{\{a\}, \{b\}, \{a, b\}\big\} \in \mathcal{P}^2(S).$$

Hence $\big(\mathcal{P}^1(S), \operatorname{curry}(\star)\big) = \big(\mathcal{P}(S), \operatorname{curry}(\star)\big)$ is a concrete curried 2-ary $(1,2)$-SuperHyperStructure.

6.10 Complex HyperStructures and SuperHyperStructures

Complex hyperstructures use set-valued operations compatible with complex scalar multiplication; complex superhyperstructures iterate powersets, enabling higher-order, complex-linear, hierarchical set-valued operations.

Let H be a complex vector space. For nonempty subsets $A, B \subseteq H$ and $\lambda \in \mathbb{C}$ define the (Minkowski) sum and scaling by

$$A + B \ := \ \{a + b \mid a \in A,\ b \in B\} \subseteq H, \qquad \lambda A \ := \ \{\lambda a \mid a \in A\} \subseteq H.$$

For convenience we also write $\mathcal{P}_*(H) := \mathcal{P}(H) \setminus \{\varnothing\}$.

For each $r \in \mathbb{N}_0$ define $\mathcal{P}^0(H) := H$ and $\mathcal{P}^{r+1}(H) := \mathcal{P}(\mathcal{P}^r(H))$. Define recursively the scalar action on $\mathcal{P}^r(H)$ by

$$\lambda \cdot x := \lambda x \quad (x \in \mathcal{P}^0(H) = H), \qquad \lambda \cdot A := \{\lambda \cdot a \mid a \in A\} \quad (A \in \mathcal{P}^{r+1}(H)).$$

(Thus, at every level, scaling is applied elementwise.)

Likewise, define recursively an "additive" operation $\oplus_r$ on $\mathcal{P}^r(H)$ by

$$x \oplus_0 y := x + y \quad (x, y \in H), \qquad A \oplus_{r+1} B := \{a \oplus_r b \mid a \in A,\ b \in B\} \quad (A, B \in \mathcal{P}^{r+1}(H)).$$

At level $r = 1$, $\oplus_1$ is exactly the Minkowski sum on subsets of H.

Definition 6.10.1 (Complex hyperoperation)**.** Let H be a complex vector space and let $k \in \mathbb{N}$. A map

$$\circ : \ H^k \longrightarrow \mathcal{P}_*(H)$$

is called a *complex (multi-)linear hyperoperation* if for every slot $i \in \{1, \dots, k\}$, all $x_1, \dots, x_k, x_i' \in H$, and all $\lambda, \mu \in \mathbb{C}$,

$$\circ(x_1, \dots, \lambda x_i + \mu x_i', \dots, x_k) \ \subseteq \ \lambda \circ (x_1, \dots, x_i, \dots, x_k) \ + \ \mu \circ (x_1, \dots, x_i', \dots, x_k),$$

where the right-hand side uses Minkowski sum and set-scaling in H.

Definition 6.10.2 (Complex Σ-hyperstructure)**.** Fix a single-sorted finitary signature $\Sigma = (\mathsf{Func}, \mathsf{Rel}, \mathrm{ar}_{\mathsf{Func}}, \mathrm{ar}_{\mathsf{Rel}})$ and a finite index set Q of hyperoperation symbols with arities $(k_q)_{q \in Q}$. A *complex Σ-hyperstructure* is a tuple

$$\mathbf{H} = \Big(H, \ (f^{\mathbf{H}})_{f \in \mathsf{Func}}, \ (R^{\mathbf{H}})_{R \in \mathsf{Rel}}, \ (\circ_{\mathbf{H}}^{(q)})_{q \in Q}\Big)$$

such that:

- H is a nonzero complex vector space (in particular $H \neq \varnothing$);
- for each $f \in \mathsf{Func}$ of arity m, the interpretation $f^{\mathbf{H}} : H^m \to H$ is $\mathbb{C}$-multilinear (or, more generally, $\mathbb{C}$-polynomial, as chosen);
- for each $q \in Q$, the hyperoperation $\circ_{\mathbf{H}}^{(q)} : H^{k_q} \to \mathcal{P}_*(H)$ is a complex multilinear hyperoperation in the sense above;
- each $R^{\mathbf{H}} \subseteq H^r$ is a relation symbol interpretation (no additional "complex" axiom is required unless desired).

Remark 6.10.3. If $Q = \varnothing$, then Definition 6.10.2 reduces to an ordinary complex Σ-algebra (a $\mathbb{C}$-linear Σ-structure). If $\Sigma = \varnothing$ and Q has a single binary symbol, one obtains a "complex semihypergroup-like" object with a $\mathbb{C}$-multilinear hyperoperation.

Example 6.10.4 (Real-world example of a complex Σ-hyperstructure)**. Coherent optical beamforming with uncertain scattering.** Consider a phased-array optical/radar system that transmits and receives narrowband complex baseband signals. Let $H = \mathbb{C}^d$ be the complex vector space of *array snapshots* (one complex amplitude per sensor), where d is the number of antenna/photodetector elements.

Fix a finitary signature Σ that contains (at least) the following function symbols:

$$\mathsf{Func} \ni \{ +, \cdot, \langle\cdot,\cdot\rangle, T \},$$

interpreted in $\mathbf{H}$ as the usual complex vector addition and scalar multiplication on $\mathbb{C}^d$, the Hermitian inner product $\langle x, y\rangle = x^* y$, and a fixed linear operator $T : H \to H$ (e.g. a steering/beamforming matrix or a propagation operator).

To model *multi-path scattering uncertainty*, introduce a finite index set Q of hyperoperation symbols. For a representative binary symbol $q \in Q$ (arity $k_q = 2$), define a complex multilinear hyperoperation

$$\circ_{\mathbf{H}}^{(q)} : H \times H \to \mathcal{P}_*(H), \qquad (x, y) \longmapsto x \circ_{\mathbf{H}}^{(q)} y,$$

by

$$x \circ_{\mathbf{H}}^{(q)} y \ := \ \Big\{ \alpha x + \beta y \ \Big| \ (\alpha, \beta) \in U \Big\},$$

where $U \subseteq \mathbb{C}^2$ is a prescribed nonempty set of admissible complex gains (e.g. gains consistent with calibration bounds, a scattering envelope, or interval/elliptic uncertainty in amplitude and phase). Then:

- H is a nonzero complex vector space;
- the Σ-functions are $\mathbb{C}$-multilinear (or polynomial) maps on H;
- $\circ_{\mathbf{H}}^{(q)}$ is a *hyper*operation producing a nonempty set of possible received superpositions, capturing ambiguity in the complex path gains;
- any additional Σ-relations (e.g. "SNR$\geq \tau$" constraints) can be interpreted as subsets of H^r.

Hence the system state space together with its linear signal-processing primitives and the uncertainty hyperoperation forms a concrete complex Σ-hyperstructure in the sense of Definition 6.10.2.

Definition 6.10.5 (Complex Θ-superhyperoperation)**.** Let H be a complex vector space and fix a profile $\Theta = (m, n, k)$ with $m, n \in \mathbb{N}_0$ and $k \in \mathbb{N}$. A map

$$\star : \ (\mathcal{P}^m(H))^k \longrightarrow \mathcal{P}^n(H)$$

is called *complex* Θ*-homogeneous* if for every $\lambda \in \mathbb{C}$ and all $A_1, \ldots, A_k \in \mathcal{P}^m(H)$,

$$\star(\lambda \cdot A_1, \ldots, \lambda \cdot A_k) \ = \ \lambda \cdot \star(A_1, \ldots, A_k),$$

where $\lambda \cdot (\cdot)$ is the lifted scalar action on $\mathcal{P}^m(H)$ and $\mathcal{P}^n(H)$.

If, in addition, for every slot i and all $\lambda, \mu \in \mathbb{C}$,

$$\star(A_1, \ldots, \lambda \cdot A_i \oplus_m \mu \cdot A_i', \ldots, A_k) \ \subseteq \ \lambda \cdot \star(A_1, \ldots, A_i, \ldots, A_k) \ \oplus_n \ \mu \cdot \star(A_1, \ldots, A_i', \ldots, A_k),$$

then $\star$ is called a *complex (multi-)linear* Θ*-superhyperoperation.*

Definition 6.10.6 (Complex Θ-superhyperstructure)**.** Fix a complex vector space $H \neq \{0\}$ and a profile $\Theta = (m, n, k)$. A *complex Θ-superhyperstructure over* H is a pair

$$\mathbf{S} = (\mathcal{P}^m(H), \star_{\mathbf{S}})$$

where $\star_{\mathbf{S}} : (\mathcal{P}^m(H))^k \to \mathcal{P}^n(H)$ is a complex Θ-homogeneous superhyperoperation. If $\star_{\mathbf{S}}$ is complex multilinear (in the sense of the previous definition), we call $\mathbf{S}$ a *complex multilinear Θ-superhyperstructure.*

Remark 6.10.7 (Consistency with lower levels)**.** If $(m, n) = (0, 1)$, then $\star : H^k \to \mathcal{P}(H)$ is exactly a hyperoperation on H, and complex multilinearity reduces to the standard inclusion-type linearity for set-valued maps. If $(m, n) = (0, 0)$, then $\star : H^k \to H$ is an ordinary k-ary operation and one recovers the usual $\mathbb{C}$-multilinearity condition.

Example 6.10.8 (Real-world example of a complex Σ-superhyperstructure)**. Robust set-based interference management in multi-user MIMO.** Consider an uplink multi-user MIMO receiver that observes complex baseband vectors in $H = \mathbb{C}^d$. At any time, each user u has an *uncertain* feasible set of received signal vectors $A_u \subseteq H$ due to unknown channel coefficients, phase noise, and bounded interference.

Fix a profile $\Theta = (m, n, k)$ with $m = n = 1$ and $k = 2$, so $\mathcal{P}^m(H) = \mathcal{P}(H)$ and $\mathcal{P}^n(H) = \mathcal{P}(H)$. Define a superhyperoperation

$$\star : \mathcal{P}(H) \times \mathcal{P}(H) \longrightarrow \mathcal{P}(H)$$

by the *robust Minkowski-type aggregation*

$$A \star B \;:=\; \Big\{ x + y \;\Big|\; x \in A,\ y \in B \Big\} \;\cap\; \Big\{ z \in H \;\Big|\; \|Wz\| \leq \tau \Big\},$$

where $W : H \to H$ is a fixed complex linear filter (e.g. a whitening or projection operator) and $\tau > 0$ is an admissible interference/power threshold. Operationally, $A \star B$ is the set of all superpositions of two uncertain users' signals that remain feasible after linear filtering and a system constraint.

This $\star$ is complex Θ-homogeneous: for $\lambda \in \mathbb{C}$,

$$(\lambda A) \star (\lambda B) = \lambda (A \star B),$$

because scalar multiplication distributes through addition and the constraint can be chosen homogeneous (e.g. $\|W(\lambda z)\| = |\lambda| \|Wz\|$ with an amplitude-scaled threshold). Moreover, $\star$ is set-linear in each slot in the sense of Definition 6.10.6 (with the lifted operations $\oplus_1$ on $\mathcal{P}(H)$).

Therefore, the set-universe $\mathcal{P}(H)$ of uncertain complex signal states equipped with $\star$ is a concrete complex Θ-superhyperstructure, modeling *set-based* robust combination of uncertain complex-valued signals under linear processing and feasibility constraints.

6.11 Dynamic HyperStructures and Dynamic SuperHyperStructures

Let $(\mathbb{T}, \leq)$ be a nonempty totally ordered (or, more generally, preordered) time set (e.g. $\mathbb{T} = \mathbb{N}$ for discrete time or $\mathbb{T} = \mathbb{R}_{\geq 0}$ for continuous time). We view $(\mathbb{T}, \leq)$ as a small category whose objects are times $\tau \in \mathbb{T}$ and with a unique arrow $\tau \to \tau'$ whenever $\tau \leq \tau'$.

Fix a single–sorted finitary signature

$$\Sigma = \big(\mathsf{Func}, \mathsf{Rel}, \mathrm{ar}_{\mathsf{Func}}, \mathrm{ar}_{\mathsf{Rel}}\big),$$

and fix a finite index set Q of designated hyperoperations (to be part of the structure), where each $q \in Q$ has arity $m_q \in \mathbb{N}$.

Definition 6.11.1 (Σ–hyperstructure and homomorphism)**.** A Σ–*hyperstructure* is a tuple

$$\mathbf{H} = \Big(|\mathbf{H}|,\ (f^{\mathbf{H}})_{f \in \mathsf{Func}},\ (R^{\mathbf{H}})_{R \in \mathsf{Rel}},\ (\circ_{\mathbf{H}}^{(q)})_{q \in Q} \Big),$$

where $|\mathbf{H}| \neq \varnothing$, $f^{\mathbf{H}} : |\mathbf{H}|^{\mathrm{ar}(f)} \to |\mathbf{H}|$ for each $f \in \mathsf{Func}$, $R^{\mathbf{H}} \subseteq |\mathbf{H}|^{\mathrm{ar}(R)}$ for each $R \in \mathsf{Rel}$, and each

$$\circ_{\mathbf{H}}^{(q)} :\ |\mathbf{H}|^{m_q} \longrightarrow \mathcal{P}\big(|\mathbf{H}|\big) \qquad (q \in Q)$$

is a hyperoperation.

Given Σ–hyperstructures $\mathbf{H}$ and $\mathbf{K}$, a map $\varphi : |\mathbf{H}| \to |\mathbf{K}|$ is a *homomorphism* if:

$$\varphi\big(f^{\mathbf{H}}(\vec{x})\big) = f^{\mathbf{K}}\big(\varphi(\vec{x})\big), \qquad \vec{x} \in R^{\mathbf{H}} \ \Rightarrow\ \varphi(\vec{x}) \in R^{\mathbf{K}},$$

and, for every $q \in Q$ and every $\vec{x} \in |\mathbf{H}|^{m_q}$,

$$\varphi\big(\circ_{\mathbf{H}}^{(q)}(\vec{x})\big) \ \subseteq\ \circ_{\mathbf{K}}^{(q)}\big(\varphi(\vec{x})\big),$$

where $\varphi(A) := \{\varphi(a) : a \in A\}$ for $A \subseteq |\mathbf{H}|$. If φ is bijective and all inclusions above are equalities (and the relation condition is equivalence), then φ is an *isomorphism*.

Definition 6.11.2 (Dynamic Σ–hyperstructure)**.** A *dynamic* Σ–*hyperstructure* is a pair

$$\mathbb{H} = \Big((\mathbf{H}_\tau)_{\tau \in \mathbb{T}},\ (\Phi_{\tau,\tau'})_{\tau \leq \tau'} \Big),$$

where:

- for each $\tau \in \mathbb{T}$, $\mathbf{H}_\tau$ is a Σ–hyperstructure;
- for each $\tau \leq \tau'$, $\Phi_{\tau,\tau'} : \mathbf{H}_\tau \to \mathbf{H}_{\tau'}$ is a homomorphism in the sense of Definition 6.11.1;
- (**identity**) $\Phi_{\tau,\tau} = \mathrm{id}_{\mathbf{H}_\tau}$ for all τ;
- (**cocycle / consistency**) if $\tau \leq \tau' \leq \tau''$, then

$$\Phi_{\tau',\tau''} \circ \Phi_{\tau,\tau'} = \Phi_{\tau,\tau''}.$$

Equivalently, a dynamic Σ–hyperstructure is a functor

$$\mathbb{H}: \ (\mathbb{T}, \leq) \longrightarrow \mathrm{HypStr}_{\Sigma,Q},$$

from the time category into the category of Σ–hyperstructures (with the chosen hyperoperations).

Remark 6.11.3 (Fixed-carrier special case)**.** Often the carrier is fixed, i.e. $|\mathbf{H}_\tau| = X$ for all τ, and $\Phi_{\tau,\tau'} = \mathrm{id}_X$. Then the dynamics is entirely in the time–dependent interpretations f_τ, R_τ, and hyperoperations $\circ_\tau^{(q)}$ on the same underlying set X.

Example 6.11.4 (Real-world example of a dynamic Σ–hyperstructure)**. Evolving chemical reaction hyperstructure under changing operating conditions.** Let $\mathbb{T}$ be a totally ordered time set (e.g. discrete days $\mathbb{T} = \mathbb{N}$). A chemical plant maintains a catalogue of *reactive species* and *reaction rules* that evolve over time because catalysts are replaced, temperature/pressure regimes are changed, or new side reactions are discovered.

For each $\tau \in \mathbb{T}$, let $\mathbf{H}_\tau$ be a Σ–hyperstructure whose carrier H_τ is the set of chemical species that are relevant/observable at time τ. Include in Σ function symbols (examples):

$$\Sigma \supseteq \{ \oplus \text{ (mixture)}, \ \mathrm{cat} \text{ (catalyst application)}, \ \mathrm{rate} \},$$

interpreted by maps $f^{\mathbf{H}_\tau} : H_\tau^m \to H_\tau$ that encode standard deterministic bookkeeping (e.g. forming a mixture label, applying a known catalyst, or evaluating a nominal rate law).

To model *uncertain multi-product outcomes*, fix a finite set Q of hyperoperation symbols and interpret, for a representative $q \in Q$ of arity $k_q = 2$, a hyperreaction map

$$\circ_\tau^{(q)} : H_\tau \times H_\tau \longrightarrow \mathcal{P}_*(H_\tau), \qquad (X, Y) \longmapsto X \circ_\tau^{(q)} Y,$$

where $X \circ_\tau^{(q)} Y$ is the nonempty set of all products that may be obtained from combining X and Y at time τ (including side products that are only possible under the current regime).

As the operating regime changes from τ to τ', there is a natural homomorphism

$$\Phi_{\tau,\tau'} : \mathbf{H}_\tau \to \mathbf{H}_{\tau'}$$

sending each species name/identifier in H_τ to its updated identifier in $H_{\tau'}$ (e.g. after merging synonyms, re-indexing, or mapping deprecated intermediates to their tracked substitutes), and preserving all Σ-operations and the hyperoperations:

$$\Phi_{\tau,\tau'}\big(f^{\mathbf{H}_\tau}(x_1, \ldots, x_m)\big) = f^{\mathbf{H}_{\tau'}}\big(\Phi_{\tau,\tau'}(x_1), \ldots, \Phi_{\tau,\tau'}(x_m)\big),$$

$$\Phi^{[*]}_{\tau,\tau'}\big(x \circ_\tau^{(q)} y\big) = \Phi_{\tau,\tau'}(x) \circ_{\tau'}^{(q)} \Phi_{\tau,\tau'}(y),$$

with the usual identity and cocycle conditions. Hence $\mathbb{H} = \big((\mathbf{H}_\tau)_{\tau\in\mathbb{T}}, (\Phi_{\tau,\tau'})_{\tau\leq\tau'}\big)$ forms a concrete dynamic Σ–hyperstructure capturing the time evolution of a reaction network with set-valued (uncertain) reaction outcomes.

Fix a *profile* $\Theta = (m, n, k)$ with $m, n \in \mathbb{N}_0$ and $k \in \mathbb{N}$.

Definition 6.11.5 (Θ–superhyperstructure)**.** A Θ–*superhyperstructure* over a base set $H \neq \varnothing$ is an object

$$\mathbf{S} = \big(\mathcal{P}^m(H),\ \star_{\mathbf{S}}\big), \qquad \star_{\mathbf{S}} : \big(\mathcal{P}^m(H)\big)^k \longrightarrow \mathcal{P}^n(H).$$

Definition 6.11.6 (Dynamic Θ–superhyperstructure)**.** A *dynamic* Θ–*superhyperstructure* is a pair

$$\mathbb{S} = \Big((\mathbf{S}_\tau)_{\tau\in\mathbb{T}},\ (\alpha_{\tau,\tau'})_{\tau\le\tau'} \Big),$$

where:

- for each $\tau \in \mathbb{T}$, $\mathbf{S}_\tau = \big(\mathcal{P}^m(H_\tau), \star_\tau\big)$ is a Θ–superhyperstructure over some nonempty base set H_τ;
- for each $\tau \le \tau'$, $\alpha_{\tau,\tau'} : H_\tau \to H_{\tau'}$ is a bijection (transport of the base);
- (**identity**) $\alpha_{\tau,\tau} = \mathrm{id}_{H_\tau}$ for all τ;
- (**cocycle**) if $\tau \le \tau' \le \tau''$, then $\alpha_{\tau,\tau''} = \alpha_{\tau',\tau''} \circ \alpha_{\tau,\tau'}$;
- (**compatibility of superhyperoperations**) for all $\tau \le \tau'$,

  $$\alpha^{[n]}_{\tau,\tau'}\Big(\star_\tau(A_1,\dots,A_k) \Big) \ = \ \star_{\tau'}\Big(\alpha^{[m]}_{\tau,\tau'}(A_1),\dots,\alpha^{[m]}_{\tau,\tau'}(A_k) \Big)$$

  for every $(A_1,\dots,A_k) \in \big(\mathcal{P}^m(H_\tau)\big)^k$.

Example 6.11.7 (Real-world example of a dynamic Θ–superhyperstructure)**. Time-varying set-based reachability in an urban logistics system.** Consider a delivery platform operating over time $\mathbb{T}$, where at each time τ the set H_τ consists of all *active states* of the fleet, e.g. a state can encode (vehicle ID, location zone, remaining capacity, battery level). Because vehicles enter/leave service, H_τ genuinely changes with τ.

Fix a profile $\Theta = (m, n, k)$ with $m = n = 1$ and $k = 2$. For each τ, define a Θ–superhyperstructure

$$\mathbf{S}_\tau = \big(\mathcal{P}(H_\tau), \star_\tau\big), \qquad \star_\tau : \mathcal{P}(H_\tau) \times \mathcal{P}(H_\tau) \to \mathcal{P}(H_\tau),$$

where, for two sets of fleet states $A, B \subseteq H_\tau$, the value $A \star_\tau B$ is interpreted as the set of states that are reachable by *combining* two concurrent sources of uncertainty at time τ:

- uncertainty from *traffic and travel-time variation* (affecting where vehicles in A can move),
- uncertainty from *dispatch/admission decisions* (affecting which tasks in B are accepted/merged).

Concretely, one may define

$$A \star_\tau B := \Big\{ h' \in H_\tau \;\Big|\; \exists h \in A,\ \exists b \in B :\ h' \in \mathrm{Step}_\tau(h,b) \Big\},$$

where $\mathrm{Step}_\tau(h,b) \subseteq H_\tau$ is the (nonempty) set of feasible next states produced by applying dispatch decision b to current state h under the time-τ traffic regime. This is naturally set-valued and depends on τ.

To compare different times, assume there is a canonical *relabeling/transport* bijection

$$\alpha_{\tau,\tau'} : H_\tau \to H_{\tau'}$$

that matches "the same" vehicle-state across times (e.g. by vehicle ID and normalized coordinates), and satisfies the cocycle conditions. Compatibility of superhyperoperations expresses *time-consistency* of the state evolution model: transporting the result of a combined reachability computation equals computing reachability after transport:

$$\alpha^{[1]}_{\tau,\tau'}\big(A \star_\tau B\big) = \alpha^{[1]}_{\tau,\tau'}(A) \ \star_{\tau'}\ \alpha^{[1]}_{\tau,\tau'}(B).$$

Therefore, $\mathbb{S} = \big((\mathbf{S}_\tau)_{\tau\in\mathbb{T}}, (\alpha_{\tau,\tau'})_{\tau\le\tau'}\big)$ is a concrete dynamic Θ–superhyperstructure modeling time-varying set-based reachability/feasibility in a real logistics platform.

For a map $\alpha : H \to K$ and an integer $r \in \mathbb{N}_0$, we use the standard *iterated powerset lift* $\alpha^{[r]} : \mathcal{P}^r(H) \to \mathcal{P}^r(K)$ defined recursively by

$$\alpha^{[0]} := \alpha, \qquad \alpha^{[r+1]} := \mathcal{P}\big(\alpha^{[r]}\big) \ \ (r \ge 0),$$

where $\mathcal{P}(g)(A) := \{g(a) : a \in A\}$ for $A \subseteq \mathrm{dom}(g)$. For k-tuples we write

$$\big(\alpha^{[m]}\big)^{\times k}(A_1,\dots,A_k) := \big(\alpha^{[m]}(A_1),\dots,\alpha^{[m]}(A_k)\big).$$

Lemma 6.11.8 (Functoriality of the lift)**.** *Let $\alpha : H \to K$ and $\beta : K \to L$ be maps. Then for every $r \in \mathbb{N}_0$,*

$$(\beta\circ\alpha)^{[r]} = \beta^{[r]} \circ \alpha^{[r]}, \qquad (\mathrm{id}_H)^{[r]} = \mathrm{id}_{\mathcal{P}^r(H)}.$$

Proof. We argue by induction on r.

Base $r = 0$. Immediate: $(\beta\circ\alpha)^{[0]} = \beta\circ\alpha = \beta^{[0]}\circ\alpha^{[0]}$ and $(\mathrm{id}_H)^{[0]} = \mathrm{id}_H = \mathrm{id}_{\mathcal{P}^0(H)}$.

Step $r \to r+1$. Assume $(\beta\circ\alpha)^{[r]} = \beta^{[r]}\circ\alpha^{[r]}$. For any $A \in \mathcal{P}^{r+1}(H) = \mathcal{P}(\mathcal{P}^r(H))$,

$$(\beta\circ\alpha)^{[r+1]}(A) = \mathcal{P}\big((\beta\circ\alpha)^{[r]}\big)(A) = \{(\beta\circ\alpha)^{[r]}(a) : a \in A\}.$$

By the induction hypothesis this equals $\{\beta^{[r]}(\alpha^{[r]}(a)) : a \in A\}$, i.e.

$$(\beta\circ\alpha)^{[r+1]}(A) = \mathcal{P}(\beta^{[r]})(\mathcal{P}(\alpha^{[r]})(A)) = \beta^{[r+1]}(\alpha^{[r+1]}(A)).$$

Hence $(\beta\circ\alpha)^{[r+1]} = \beta^{[r+1]}\circ\alpha^{[r+1]}$. The identity statement is analogous. □

Lemma 6.11.9 (Lifts of bijections)**.** *If $\alpha : H \to K$ is a bijection, then $\alpha^{[r]} : \mathcal{P}^r(H) \to \mathcal{P}^r(K)$ is a bijection for all $r \in \mathbb{N}_0$, and*

$$(\alpha^{-1})^{[r]} = \big(\alpha^{[r]}\big)^{-1}.$$

Proof. For $r = 0$ the claim is clear. Assume $\alpha^{[r]}$ is a bijection with inverse $(\alpha^{-1})^{[r]}$. Then $\alpha^{[r+1]} = \mathcal{P}(\alpha^{[r]})$ is a bijection with inverse $\mathcal{P}((\alpha^{[r]})^{-1}) = \mathcal{P}((\alpha^{-1})^{[r]}) = (\alpha^{-1})^{[r+1]}$. (The equality uses Lemma 6.11.8 applied to $\alpha^{-1} \circ \alpha = \mathrm{id}$.) □

Fix a profile $\Theta = (m, n, k)$.

Definition 6.11.10 (Category of Θ–superhyperstructures)**.** Define a category $\mathrm{SuHyStr}_\Theta$ as follows.

- Objects are Θ–superhyperstructures $\mathbf{S} = (\mathcal{P}^m(H), \star_{\mathbf{S}})$.
- A morphism $\alpha : \mathbf{S} \to \mathbf{T}$, where $\mathbf{S} = (\mathcal{P}^m(H), \star)$ and $\mathbf{T} = (\mathcal{P}^m(K), \diamond)$, is a bijection $\alpha : H \to K$ such that for all $(A_1, \dots, A_k) \in (\mathcal{P}^m(H))^k$,

$$\alpha^{[n]}\big(\star(A_1, \dots, A_k)\big) = \diamond\Big(\big(\alpha^{[m]}\big)^{\times k}(A_1, \dots, A_k)\Big).$$

Proposition 6.11.11 (Dynamic Θ–superhyperstructures are functors)**.** *A dynamic Θ–superhyperstructure $\mathbb{S} = \big((\mathbf{S}_\tau)_{\tau\in\mathbb{T}}, (\alpha_{\tau,\tau'})_{\tau\le\tau'}\big)$ in the sense of Definition 6.11.6 is equivalently a functor*

$$\mathbb{S} : (\mathbb{T}, \le) \longrightarrow \mathrm{SuHyStr}_\Theta, \qquad \tau \longmapsto \mathbf{S}_\tau, \quad (\tau \le \tau') \longmapsto \alpha_{\tau,\tau'}.$$

Proof. By Definition 6.11.6, each $\alpha_{\tau,\tau'} : H_\tau \to H_{\tau'}$ is a bijection, hence gives a candidate morphism in $\mathrm{SuHyStr}_\Theta$. The compatibility axiom of Definition 6.11.6 is exactly the morphism condition in Definition 6.11.10. The identity and cocycle axioms for $\alpha_{\tau,\tau'}$ are precisely the functoriality conditions for arrows in the preorder category $(\mathbb{T}, \le)$. Conversely, any such functor provides the same data and satisfies the same axioms by unpacking definitions. □

Proposition 6.11.12 (Transport of a superhyperoperation along a bijection)**.** *Let $\mathbf{S} = (\mathcal{P}^m(H), \star)$ be a Θ–superhyperstructure and let $\alpha : H \to K$ be a bijection. Define an operation $\diamond : (\mathcal{P}^m(K))^k \to \mathcal{P}^n(K)$ by*

$$\diamond := \alpha^{[n]} \circ \star \circ \Big(\big((\alpha^{-1})^{[m]}\big)^{\times k}\Big).$$

Then $\mathbf{T} := (\mathcal{P}^m(K), \diamond)$ is a Θ–superhyperstructure and $\alpha : \mathbf{S} \to \mathbf{T}$ is an isomorphism in $\mathrm{SuHyStr}_\Theta$. Moreover, $\diamond$ is the unique *operation on K making α an isomorphism.*

Proof. Well-definedness is clear because $\alpha^{[n]}$ and $(\alpha^{-1})^{[m]}$ are well-defined maps on iterated powersets. To check that α is a morphism $\mathbf{S} \to \mathbf{T}$, take $(A_1, \dots, A_k) \in (\mathcal{P}^m(H))^k$. Then, by the definition of $\diamond$ and Lemma 6.11.9,

$$\diamond\Big(\big(\alpha^{[m]}\big)^{\times k}(A_1, \dots, A_k)\Big) = \alpha^{[n]}\Big(\star\Big(\big((\alpha^{-1})^{[m]}\big)^{\times k}\big(\alpha^{[m]}(A_1), \dots, \alpha^{[m]}(A_k)\big)\Big)\Big).$$

But $(\alpha^{-1})^{[m]} \circ \alpha^{[m]} = \mathrm{id}_{\mathcal{P}^m(H)}$, hence the inner k-tuple reduces to $(A_1, \dots, A_k)$. Therefore

$$\diamond\Big(\big(\alpha^{[m]}\big)^{\times k}(A_1, \dots, A_k)\Big) = \alpha^{[n]}\big(\star(A_1, \dots, A_k)\big),$$

which is exactly the morphism condition. Bijectivity of α plus equality (not mere inclusion) shows it is an isomorphism.

For uniqueness, suppose $\diamond'$ also makes α a morphism, i.e. $\alpha^{[n]}(\star(\vec{A})) = \diamond'((\alpha^{[m]})^{\times k}(\vec{A}))$ for all $\vec{A}$. Given any $\vec{B} \in (\mathcal{P}^m(K))^k$, set $\vec{A} := ((\alpha^{-1})^{[m]})^{\times k}(\vec{B})$. Then $(\alpha^{[m]})^{\times k}(\vec{A}) = \vec{B}$ and hence

$$\diamond'(\vec{B}) = \alpha^{[n]}(\star(\vec{A})) = \alpha^{[n]} \circ \star \circ \big(((\alpha^{-1})^{[m]})^{\times k}\big)(\vec{B}) = \diamond(\vec{B}).$$

Thus $\diamond' = \diamond$. □

Theorem 6.11.13 (Rigidity: dynamics is "constant up to relabeling")**.** *Let $\mathbb{S} = \big((\mathbf{S}_\tau)_\tau, (\alpha_{\tau,\tau'})_{\tau \le \tau'}\big)$ be a dynamic Θ–superhyperstructure. Fix a reference time $\tau_0 \in \mathbb{T}$. Then for every $\tau \in \mathbb{T}$ with $\tau_0 \le \tau$, the operation $\star_\tau$ is determined by $\star_{\tau_0}$ via conjugation:*

$$\star_\tau = \alpha^{[n]}_{\tau_0,\tau} \circ \star_{\tau_0} \circ \Big(\big((\alpha^{-1}_{\tau_0,\tau})^{[m]}\big)^{\times k}\Big).$$

Equivalently, if we transport all operations back to the fixed base set H_{τ_0} by

$$\widehat{\star}_\tau := (\alpha^{-1}_{\tau_0,\tau})^{[n]} \circ \star_\tau \circ \Big(\big(\alpha^{[m]}_{\tau_0,\tau}\big)^{\times k}\Big),$$

then $\widehat{\star}_\tau = \star_{\tau_0}$ for all $\tau \ge \tau_0$.

Proof. Apply the compatibility axiom of Definition 6.11.6 to the pair (τ_0, τ): for every $\vec{A} \in (\mathcal{P}^m(H_{\tau_0}))^k$,

$$\alpha^{[n]}_{\tau_0,\tau}\big(\star_{\tau_0}(\vec{A})\big) = \star_\tau\big((\alpha^{[m]}_{\tau_0,\tau})^{\times k}(\vec{A})\big).$$

Now let $\vec{B} \in (\mathcal{P}^m(H_\tau))^k$ be arbitrary and set $\vec{A} := ((\alpha^{-1}_{\tau_0,\tau})^{[m]})^{\times k}(\vec{B})$. Since $\alpha^{[m]}_{\tau_0,\tau}$ is a bijection (Lemma 6.11.9), we have $(\alpha^{[m]}_{\tau_0,\tau})^{\times k}(\vec{A}) = \vec{B}$, hence

$$\star_\tau(\vec{B}) = \alpha^{[n]}_{\tau_0,\tau}\big(\star_{\tau_0}(\vec{A})\big) = \alpha^{[n]}_{\tau_0,\tau} \circ \star_{\tau_0} \circ \Big(\big((\alpha^{-1}_{\tau_0,\tau})^{[m]}\big)^{\times k}\Big)(\vec{B}),$$

which is the stated conjugation formula. The "transport back" statement follows by composing both sides with $(\alpha^{-1}_{\tau_0,\tau})^{[n]}$. □

Proposition 6.11.14 (Construction on $\mathbb{T} = \mathbb{N}$ from one-step data)**.** *Assume $\mathbb{T} = \mathbb{N}$ with the usual order. Fix nonempty sets $(H_t)_{t\in\mathbb{N}}$, bijections $\alpha_{t,t+1} : H_t \to H_{t+1}$, and an initial Θ–superhyperstructure $\mathbf{S}_0 = (\mathcal{P}^m(H_0), \star_0)$. Define recursively operations $\star_{t+1}$ on H_{t+1} by*

$$\star_{t+1} := \alpha^{[n]}_{t,t+1} \circ \star_t \circ \Big(\big((\alpha^{-1}_{t,t+1})^{[m]}\big)^{\times k}\Big).$$

For $t \le t'$, define $\alpha_{t,t'} := \alpha_{t'-1,t'} \circ \cdots \circ \alpha_{t,t+1}$ and $\alpha_{t,t} := \mathrm{id}$. Then $\mathbb{S} = \big((\mathbf{S}_t)_{t\in\mathbb{N}}, (\alpha_{t,t'})_{t\le t'}\big)$ is a dynamic Θ–superhyperstructure.

Proof. By construction each $\alpha_{t,t+1}$ is bijective, hence so is each composite $\alpha_{t,t'}$; identity and cocycle properties hold by definition of composition.

It remains to verify compatibility. For $t' = t + 1$, compatibility holds by the defining formula of $\star_{t+1}$, since that formula is exactly the condition that $\alpha_{t,t+1}$ is a morphism $\mathbf{S}_t \to \mathbf{S}_{t+1}$ (cf. Proposition 6.11.12).

For general $t \le t'$, we proceed by induction on $(t' - t)$. Assume compatibility holds for (t, t'). Then for $(t, t' + 1)$ and any $\vec{A} \in (\mathcal{P}^m(H_t))^k$,

$$\alpha^{[n]}_{t,t'+1}(\star_t(\vec{A})) = \alpha^{[n]}_{t',t'+1}\big(\alpha^{[n]}_{t,t'}(\star_t(\vec{A}))\big) = \alpha^{[n]}_{t',t'+1}\Big(\star_{t'}\big((\alpha^{[m]}_{t,t'})^{\times k}(\vec{A})\big)\Big),$$

where we used Lemma 6.11.8 and the induction hypothesis. Applying one-step compatibility for $(t', t' + 1)$ yields

$$\alpha^{[n]}_{t',t'+1}\Big(\star_{t'}\big((\alpha^{[m]}_{t,t'})^{\times k}(\vec{A})\big)\Big) = \star_{t'+1}\Big((\alpha^{[m]}_{t',t'+1})^{\times k}\big((\alpha^{[m]}_{t,t'})^{\times k}(\vec{A})\big)\Big).$$

Finally, $(\alpha^{[m]}_{t',t'+1}) \circ (\alpha^{[m]}_{t,t'}) = (\alpha_{t,t'+1})^{[m]}$ by Lemma 6.11.8, hence the right-hand side equals $\star_{t'+1}((\alpha^{[m]}_{t,t'+1})^{\times k}(\vec{A}))$. Thus compatibility holds for $(t, t' + 1)$, completing the induction. □

6.12 Mixed HyperStructure and Mixed SuperHyperStructures

Mixed hyperstructures are type-aware hyperalgebraic systems: they consist of several underlying carriers (or "sorts") together with hyperoperations that accept inputs from specified sorts and return a set of outputs in a (possibly different) sort. Mixed superhyperstructures extend this idea by allowing the carriers themselves to be iterated powerset levels (nested collections), and by using superhyperoperations that map tuples of such higher-level objects to higher-level, nested set-valued outputs, again respecting the prescribed typing across sorts [400].

Definition 6.12.1 (Typed hyperprofiles)**.** [400] Let T be a nonempty set (of *types*). A *typed family of base sets* is a family $(S_a)_{a\in T}$ of nonempty sets indexed by T. For $k \ge 1$, a *typed hyperprofile* is a pair

$$\lambda = \big((a_1, \dots, a_k) \mid b\big) \in T^k \times T.$$

The arity of λ is k.

Definition 6.12.2 (Mixed HyperStructure)**.** [400] Let T be a nonempty set of types and $(S_a)_{a\in T}$ a family of nonempty base sets. Fix an index set Λ of typed hyperprofiles. A *Mixed HyperStructure* on $(S_a)_{a\in T}$ is a pair

$$H_{\mathrm{mix}} = \Big((S_a)_{a\in T},\ (\circ_\lambda)_{\lambda\in\Lambda}\Big),$$

where for each profile $\lambda = \big((a_1, \dots, a_k) \mid b\big) \in \Lambda$ the associated *mixed hyperoperation* is a map

$$\circ_\lambda : \mathrm{PS}^1(S_{a_1}) \times \cdots \times \mathrm{PS}^1(S_{a_k}) \longrightarrow \mathrm{PS}^1(S_b).$$

(Optionally, one may replace PS^1 everywhere by $(\mathcal{P}_*)^1$ to forbid empty inputs/outputs.) No further axioms (associativity, identities, etc.) are imposed unless stated.

Homogeneous case. If $T = \{a\}$ is a singleton, then $\circ_\lambda : (\mathrm{PS}(S_a))^k \to \mathrm{PS}(S_a)$ is a usual subset-level hyperoperation. Restricting each input to a singleton yields an induced k-ary hyperoperation $\widetilde{\circ}_\lambda : S_a^k \to \mathrm{PS}(S_a)$ given by

$$\widetilde{\circ}_\lambda(x_1, \dots, x_k) := \circ_\lambda(\{x_1\}, \dots, \{x_k\}).$$

Example 6.12.3 (Mixed HyperStructure: targeted promotions in an e-commerce platform)**.** Let the set of types be $T = \{\mathsf{U}, \mathsf{I}, \mathsf{C}\}$, where S_{U} is the set of user IDs, S_{I} the set of item IDs, and S_{C} the set of coupon codes. Assume we are given two predicates:

$$\mathrm{Eligible}(u, c) \in \{\mathrm{true}, \mathrm{false}\} \quad (u \in S_{\mathsf{U}},\ c \in S_{\mathsf{C}}),$$

$$\mathrm{Applies}(c, i) \in \{\mathrm{true}, \mathrm{false}\} \quad (c \in S_{\mathsf{C}},\ i \in S_{\mathsf{I}}),$$

encoding (i) whether user u may use coupon c, and (ii) whether coupon c applies to item i.

Define a mixed hyperoperation of profile $\lambda = ((\mathsf{U}, \mathsf{I}) \mid \mathsf{C})$ by

$$\circ_\lambda : \mathrm{PS}^1(S_{\mathsf{U}}) \times \mathrm{PS}^1(S_{\mathsf{I}}) \longrightarrow \mathrm{PS}^1(S_{\mathsf{C}}),$$

$$\circ_\lambda(U', I') := \Big\{ c \in S_{\mathsf{C}} \;\Big|\; \exists u \in U' \ \mathrm{Eligible}(u, c) \ \wedge\ \exists i \in I' \ \mathrm{Applies}(c, i) \Big\}.$$

Thus, given a *segment* of users U' and a *basket/campaign* item-set I', the output is the set of all coupons that are usable by at least one targeted user and applicable to at least one targeted item.

Optionally, one can include a second typed operation (profile $((\mathsf{I}) \mid \mathsf{U})$) that maps an item-set to a likely-to-buy user cohort, e.g.

$$\circ_{\mathrm{aud}} : \mathrm{PS}^1(S_{\mathsf{I}}) \to \mathrm{PS}^1(S_{\mathsf{U}}), \qquad \circ_{\mathrm{aud}}(I') := \{\, u \in S_{\mathsf{U}} \mid \exists i \in I' \ \mathrm{Interested}(u, i) \,\}.$$

Together, these typed set-valued maps constitute a concrete Mixed HyperStructure in the sense of Definition 6.12.2.

Definition 6.12.4 (Typed superhyperprofiles (depth N))**.** [400] Fix a depth $N \in \mathbb{N}$. A *typed superhyperprofile of depth N* is a datum

$$\mu = \big((a_1, d_1), \ldots, (a_k, d_k) \mid (b, e)\big),$$

where $k \geq 1$, $a_i, b \in T$, and $d_i, e \in \{0, 1, \ldots, N\}$. Define its domain and codomain by

$$\mathrm{Dom}(\mu) := \mathrm{PS}^{d_1}(S_{a_1}) \times \cdots \times \mathrm{PS}^{d_k}(S_{a_k}), \qquad \mathrm{Cod}(\mu) := \mathrm{PS}^{e}(S_b).$$

Definition 6.12.5 (Mixed SuperHyperStructure (depth N))**.** [400] Let T be a nonempty set of types and $(S_a)_{a \in T}$ a family of nonempty base sets. Fix a depth $N \in \mathbb{N}$ and an index set M of typed superhyperprofiles of depth N. A *Mixed SuperHyperStructure of depth N* on $(S_a)_{a \in T}$ is a pair

$$SH_{\mathrm{mix}} = \Big((S_a)_{a \in T},\ (\,?_\mu\,)_{\mu \in M}\Big),$$

where for each $\mu = \big((a_1, d_1), \ldots, (a_k, d_k) \mid (b, e)\big) \in M$ the associated *mixed superhyperoperation* is a map

$$?_\mu : \mathrm{Dom}(\mu) \longrightarrow \mathrm{Cod}(\mu) \quad \Big(\text{i.e. } ?_\mu : \mathrm{PS}^{d_1}(S_{a_1}) \times \cdots \times \mathrm{PS}^{d_k}(S_{a_k}) \to \mathrm{PS}^{e}(S_b)\Big).$$

(Optionally, replace PS^d everywhere by $(\mathcal{P}_*)^d$ to exclude empty sets at all levels.)

Remark 6.12.6 (Recovering Mixed HyperStructures as a special case)**.** If $N \geq 1$ and one restricts to profiles with $d_1 = \cdots = d_k = e = 1$, then a Mixed SuperHyperStructure yields (exactly) a Mixed HyperStructure signature:

$$?_\mu : (\mathrm{PS}^1(S_{a_1}) \times \cdots \times \mathrm{PS}^1(S_{a_k})) \to \mathrm{PS}^1(S_b).$$

In particular, Mixed HyperStructures embed into Mixed SuperHyperStructures by viewing them as depth-N systems supported only on level 1.

Example 6.12.7 (Mixed SuperHyperStructure (depth $N = 2$): generating candidate bundles per user)**.** Let $T = \{\mathsf{U}, \mathsf{I}\}$ with S_U a set of user IDs and S_I a set of items. Fix depth $N = 2$ and consider the typed superhyperprofile

$$\mu = \big((\mathsf{U}, 0), (\mathsf{I}, 1) \mid (\mathsf{I}, 2)\big),$$

so that

$$\mathrm{Dom}(\mu) = \mathrm{PS}^0(S_\mathsf{U}) \times \mathrm{PS}^1(S_\mathsf{I}) = S_\mathsf{U} \times \mathrm{PS}(S_\mathsf{I}), \qquad \mathrm{Cod}(\mu) = \mathrm{PS}^2(S_\mathsf{I}) = \mathrm{PS}\big(\mathrm{PS}(S_\mathsf{I})\big).$$

Assume a score function $s : S_\mathsf{U} \times S_\mathsf{I} \to \mathbb{R}$ (preference/utility) and a price map $p : S_\mathsf{I} \to \mathbb{R}_{\geq 0}$, and a budget $B(u) \geq 0$ for each user u. Define the mixed superhyperoperation

$$\Psi_\mu : S_\mathsf{U} \times \mathrm{PS}(S_\mathsf{I}) \longrightarrow \mathrm{PS}^2(S_\mathsf{I})$$

by

$$\Psi_\mu(u, A) := \Big\{ X \subseteq A \;\Big|\; 1 \leq |X| \leq 3,\ \sum_{i \in X} p(i) \leq B(u), \\ \text{and } X \text{ is among the top-}K \text{ bundles by } \sum_{i \in X} s(u, i) \Big\}.$$

Hence $\Psi_\mu(u, A)$ is a *set of bundles*, and each bundle is itself a set of items; therefore $\Psi_\mu(u, A) \in \mathrm{PS}^2(S_\mathsf{I})$ as required. In words: given a user u and an available item set A, the output is a *nested* collection of the best few admissible bundles (small, within-budget, high-score) to recommend.

Taking $M = \{\mu\}$ and the single operation Ψ_μ yields a concrete Mixed SuperHyperStructure of depth 2 in the sense of Definition 6.12.5.

Chapter 7

HyperStructure and SuperHyperStructure in Basic Science

HyperStructures and SuperHyperStructures are also studied and applied in fields beyond mathematics. In this chapter, we revisit these notions in the context of the basic sciences. As a reference, Table 7.1 summarizes classical concepts and their hyper/superhyper counterparts in chemical, physical, and medical sciences.

7.1 Chemical HyperStructure and SuperHyperStructure

Chemical Hyperstructure is an algebraic hyperstructure modeling chemical species' redox interactions, where hyperoperation maps reactant pairs to major product sets by maximal EMF [401–404].

Definition 7.1.1 (Chemical Hyperstructure)**.** [401] Let

$$G = \{ X_1, X_2, \ldots, X_k \}$$

be the finite set of chemical species (e.g. oxidation states of a given element), and let $\mathcal{P}^*(G)$ denote the set of all nonempty subsets of G. Suppose that for each ordered pair (X_i, X_j) there is a collection of possible redox reactions

$$X_i + X_j \longrightarrow Y + Z \quad \text{with standard reduction potentials } E(Y),\, E(Z)\,,$$

and let

$$\Delta E(X_i, X_j; Y, Z) \;=\; E(Y) + E(Z)$$

be the sum of the two half–cell potentials (in volts). Define a hyperoperation

$$\oplus : G \times G \longrightarrow \mathcal{P}^*(G)$$

by

$$X_i \oplus X_j \;=\; \big\{ Y,\, Z \in G \;\big|\; Y + Z \text{ arises from a spontaneous reaction}$$
$$X_i + X_j \to Y + Z \text{ with maximal } \Delta E(X_i, X_j; Y, Z) \big\}.$$

Table 7.1: Classical concepts and their hyper/superhyper counterparts in chemical, physical, and medical sciences.

Classical Concepts	**HyperConcepts**	**SuperHyperConcepts**
Chemical Structure (reaction/interaction structure)	Chemical HyperStructure	Chemical SuperHyperStructure
Chemical Graph (reaction graph, molecular graph)	Chemical HyperGraph	Chemical SuperHyperGraph
Physical Structure (system/interaction structure)	Physical HyperStructure	Physical SuperHyperStructure
Medical Structure (clinical/biological interaction structure)	Medical HyperStructure	Medical SuperHyperStructure

That is, among all possible products of the redox reaction $X_i + X_j \to Y + Z$, one selects those Y, Z for which ΔE is strictly maximal and positive. Then the pair

$$\big(G,\ \oplus\big)$$

is called a *chemical hyperstructure* on G under redox combination. If moreover $(G, \oplus)$ satisfies the associativity (resp. weak associativity) and reproduction axioms of classical hypergroups (resp. Hv–groups), one speaks of a *chemical hypergroup* (resp. *chemical Hv–group*).

Example 7.1.2 (A toy chemical hyperstructure on three "species")**.** Let $G = \{X_1, X_2, X_3\}$ be three abstract chemical species (e.g. oxidation states of an element). Assume that, for each ordered pair (X_i, X_j), there are exactly two admissible product pairs $Y + Z$, and that the summed potentials $\Delta E(X_i, X_j; Y, Z)$ have been assessed as follows:

(X_i, X_j)	candidate products (Y, Z)	ΔE
(X_1, X_2)	$(X_1, X_3),\ (X_2, X_3)$	$0.10,\ 0.25$
(X_1, X_3)	$(X_1, X_2),\ (X_3, X_3)$	$0.05,\ -0.10$
(X_2, X_3)	$(X_1, X_2),\ (X_1, X_3)$	$0.30,\ 0.15$

Following Definition 7.1.1, for each (X_i, X_j) we select the products with *strictly maximal and positive* ΔE. Thus we obtain the hyperoperation $\oplus : G \times G \to \mathcal{P}^*(G)$:

$$X_1 \oplus X_2 = \{X_2, X_3\} \qquad \text{(since 0.25 is maximal and positive)},$$

$$X_1 \oplus X_3 = \{X_1, X_2\} \qquad \text{(since 0.05 is maximal and positive)},$$

$$X_2 \oplus X_3 = \{X_1, X_2\} \qquad \text{(since 0.30 is maximal and positive)},$$

and define the remaining values (e.g. $X_i \oplus X_i$) similarly from the available reaction list. Each output is nonempty by construction, so $(G, \oplus)$ is a chemical hyperstructure in the sense of Definition 7.1.1.

A weak n-Superhyperstructure (SHv-Structure) equips the n-th powerset of a base set with a superhyperoperation satisfying weak associativity across hierarchical levels.

Definition 7.1.3 (Weak n^{th} Superhyperstructure)**.** Let S be a nonempty set and let $\mathcal{P}^n(S)$ be its n^{th} powerset. Denote by $\mathcal{P}^*(\mathcal{P}^n(S)) = \mathcal{P}(\mathcal{P}^n(S)) \setminus \{\varnothing\}$ the collection of all nonempty subsets of $\mathcal{P}^n(S)$. A *weak n-Superhyperstructure* is a pair

$$\big(\mathcal{P}^n(S),\ \circ\big)$$

where

$$\circ : \ \mathcal{P}^n(S) \times \mathcal{P}^n(S) \ \longrightarrow \ \mathcal{P}^*(\mathcal{P}^n(S))$$

is a *superhyperoperation* satisfying the *weak associativity* condition:

$$X \circ \big(Y \circ Z\big) \ \cap \ \big(X \circ Y\big) \circ Z \ \neq \ \varnothing \quad \text{for all } X, Y, Z \in \mathcal{P}^n(S).$$

If moreover for each $A \in \mathcal{P}^n(S)$ one has $A \circ \mathcal{P}^n(S) = \mathcal{P}^n(S) = \mathcal{P}^n(S) \circ A$, then $(\mathcal{P}^n(S), \circ)$ is called an *Hv–group of order n*.

Example 7.1.4 (A weak 1-Superhyperstructure via "either union or intersection")**.** Let $S = \{a, b\}$ and take $n = 1$. Then $\mathcal{P}^1(S) = \mathcal{P}(S)$. Define a superhyperoperation

$$\circ : \mathcal{P}(S) \times \mathcal{P}(S) \longrightarrow \mathcal{P}^*\big(\mathcal{P}(S)\big)$$

by

$$X \circ Y \ := \ \{\, X \cup Y, \ X \cap Y \,\} \qquad (X, Y \subseteq S).$$

Each $X \circ Y$ is a nonempty subset of $\mathcal{P}(S)$, so $\circ$ is well-defined.

We verify *weak associativity*: for any $X, Y, Z \subseteq S$, the set $Y \circ Z = \{Y \cup Z, \ Y \cap Z\}$, hence

$$X \circ (Y \circ Z) = \bigcup_{W \in Y \circ Z} (X \circ W) = (X \circ (Y \cup Z)) \ \cup \ (X \circ (Y \cap Z)).$$

In particular, $X \cup Y \cup Z$ belongs to $X \circ (Y \cup Z) \subseteq X \circ (Y \circ Z)$. Similarly, $(X \circ Y) \circ Z$ contains $X \cup Y \cup Z$ because $X \cup Y \in X \circ Y$ and then

$$(X \cup Y) \cup Z = X \cup Y \cup Z \in (X \circ Y) \circ Z$$

. Therefore

$$X \circ (Y \circ Z) \ \cap \ (X \circ Y) \circ Z \ \neq \ \varnothing,$$

indeed the intersection contains $X \cup Y \cup Z$. Hence $\big(\mathcal{P}(S), \circ\big)$ is a weak 1-Superhyperstructure in the sense of Definition 7.1.3.

7.2 Chemical HyperGraph and SuperHyperGraph

Chemical structure and reactivity are naturally multiscale. Atoms combine into bonds; bonds form functional groups; functional groups assemble into larger motifs and molecules; and, at an even higher level, molecules participate in assemblies and reaction networks. To model such nested organization in a single combinatorial object, we use *chemical superhypergraphs*, which allow successive layers of set-valued entities and may carry quantitative annotations (e.g., bond strengths, interaction energies, or reaction rates).

Definition 7.2.1 (Chemical graph)**.** [405] Let $\mathcal{A}$ be a set of atom labels (e.g., element symbols with optional tags), and let $\mathcal{B}$ be a set of bond labels (e.g., single/double/triple/aromatic, or a bond order). A *chemical graph* is a labeled undirected graph

$$G_{\text{chem}} = (V, E, \lambda_V, \lambda_E),$$

where:

- (V, E) is a finite simple undirected graph (unless explicitly allowing multiedges/loops);
- $\lambda_V : V \to \mathcal{A}$ assigns an atom label to each vertex (each $v \in V$ represents an atom);
- $\lambda_E : E \to \mathcal{B}$ assigns a bond label to each edge (each $e \in E$ represents a chemical bond).

Example 7.2.2 (A chemical graph for water H_2O)**.** Let

$$V = \{v_O, v_{H_1}, v_{H_2}\}$$

and

$$E = \{\{v_O, v_{H_1}\}, \{v_O, v_{H_2}\}\}$$

. Take atom labels $\mathcal{A} = \{\mathrm{H}, \mathrm{O}\}$ and bond labels $\mathcal{B} = \{\mathsf{single}\}$. Define

$$\lambda_V(v_O) = \mathrm{O}, \qquad \lambda_V(v_{H_1}) = \mathrm{H},$$
$$\lambda_V(v_{H_2}) = \mathrm{H},$$

and

$$\lambda_E(\{v_O, v_{H_1}\}) = \mathsf{single}, \qquad \lambda_E(\{v_O, v_{H_2}\}) = \mathsf{single}.$$

Then $G_{\mathrm{chem}} = (V, E, \lambda_V, \lambda_E)$ is a labeled simple graph representing the molecule H_2O in the sense of Definition 7.2.1.

Definition 7.2.3 (Chemical hypergraph)**.** [406–408] A *chemical hypergraph* is a triple

$$H_{\mathrm{chem}} = (V_0, E_S, E_N),$$

where:

- V_0 is a finite set of *atoms* (vertices);
- $E_S \subseteq \mathcal{P}_*(V_0)$ is a finite family of *simple hyperedges* (e.g., bonds or local motifs), so each $e \in E_S$ is a nonempty subset of V_0;
- $E_N \subseteq \mathcal{P}_*(E_S)$ is a finite family of *nesting hyperedges* (molecules), where each $e_N \in E_N$ is a nonempty subset of E_S indicating which level-1 units constitute that molecule.

Thus H_{chem} models atoms at level 0, bonds/motifs at level 1, and molecules as nonempty collections of level-1 units.

Example 7.2.4 (A chemical hypergraph with bonds and a molecule)**.** Let $V_0 = \{v_O, v_{H_1}, v_{H_2}\}$ be atomic vertices. Define the simple hyperedges (bonds)

$$E_S = \{e_1, e_2\}, \qquad e_1 = \{v_O, v_{H_1}\}, \quad e_2 = \{v_O, v_{H_2}\}.$$

Define one nesting hyperedge (the molecule) by

$$E_N = \{m\}, \qquad m = \{e_1, e_2\} \ \in \ \mathcal{P}_*(E_S).$$

Then

$$H_{\mathrm{chem}} = (V_0, E_S, E_N)$$

is a chemical hypergraph modeling atoms at level 0, bonds at level 1, and the water molecule as a collection of bonds, as in Definition 7.2.3.

Definition 7.2.5 (Chemical SuperHyperGraph)**.** Fix integers $m \geq 1$ and $n \geq 0$. A *chemical* (m,n)*-SuperHyperGraph* is a tuple

$$\mathcal{CSH}^{(m,n)} = \big(V_0,\, E_1,\, E_2,\, \ldots,\, E_m,\, w_1,\, w_2,\, \ldots,\, w_m\big),$$

satisfying:

(i) V_0 is a finite set of *atoms*.

(ii) $E_1 \subseteq \mathcal{P}_*^{\,n}(V_0)$ is a finite family of first-level chemical units (e.g., bonds, local motifs, or functional groups), represented as nonempty elements of $\mathcal{P}^{\,n}(V_0)$.

(iii) For each $\ell = 2, \ldots, m$, one has $E_\ell \subseteq \mathcal{P}_*(E_{\ell-1})$, i.e., each level-$\ell$ unit is a nonempty collection of level-$(\ell-1)$ units (e.g., substructures, molecules, and higher-order assemblies).

(iv) For each $\ell = 1, \ldots, m$, the weight map $w_\ell : E_\ell \to \mathbb{R}_{\geq 0}$ assigns a nonnegative weight (e.g., bond strength, interaction energy, stability score, or another quantitative attribute at level ℓ).

Example 7.2.6 (A chemical $(2,0)$-SuperHyperGraph with weights)**.** Take $m = 2$ and $n = 0$. Let the atom set be

$$V_0 = \{v_O, v_{H_1}, v_{H_2}\}.$$

Since $n = 0$, we have $\mathcal{P}_*^{\,0}(V_0) = V_0$, and level-1 units are nonempty subsets of V_0. Define

$$E_1 = \{e_1, e_2\},$$

$$e_1 = \{v_O, v_{H_1}\}, \quad e_2 = \{v_O, v_{H_2}\}.$$

Define one level-2 aggregate (the molecule) by

$$E_2 = \{M\}, \qquad M = \{e_1, e_2\} \;\in\; \mathcal{P}_*(E_1).$$

Assign nonnegative weights (e.g. bond strengths and molecular stability):

$$w_1(e_1) = 1, \quad w_1(e_2) = 1, \qquad w_2(M) = 2.$$

Then

$$\mathcal{CSH}^{(2,0)} = \big(V_0, E_1, E_2, w_1, w_2\big)$$

is a concrete chemical (m,n)-SuperHyperGraph in the sense of Definition 7.2.5.

7.3 Physical HyperStructure and SuperHyperStructure

Physical hyperstructure models interactions among physical states via set-valued outcomes, constrained by conservation laws and symmetry covariance conditions [409]. Physical superhyperstructure lifts those interactions to nested state-collections, enforcing feasibility and covariance on atoms via flattening across hierarchy levels [409].

Definition 7.3.1 (Physical HyperStructure)**.** [409] Let $S_{\text{phy}} \neq \varnothing$ be a set of physical states. Let $(V, +)$ be a commutative monoid (e.g., $V = \mathbb{R}^d$ with vector addition) encoding conserved or controlled quantities (momentum, energy, charge, etc.). Let $\nu : S_{\text{phy}} \to V$ be a valuation map. Let G be a symmetry group acting on S_{phy}, and let $\rho : G \to \mathrm{Aut}(V)$ be a representation such that

$$\nu(g \cdot s) = \rho(g)\,\nu(s) \qquad (g \in G,\ s \in S_{\text{phy}}).$$

Fix a signature Σ_{phy} of interaction symbols; each $\sigma \in \Sigma_{\text{phy}}$ has arity $\mathrm{ar}(\sigma) \geq 1$. For each $\sigma \in \Sigma_{\text{phy}}$ choose:

$D_\sigma \subseteq S_{\text{phy}}^{\mathrm{ar}(\sigma)}$ (feasibility domain), $C_\sigma \subseteq V$ (closed convex cone of admissible net losses/gains).

A *Physical HyperStructure* is a tuple

$$\mathrm{PHS} = \Big(S_{\text{phy}},\ \Sigma_{\text{phy}},\ (\sigma_S)_{\sigma\in\Sigma_{\text{phy}}},\ \nu,\ G,\ (C_\sigma)_{\sigma\in\Sigma_{\text{phy}}}\Big),$$

where each interaction hyperoperation

$$\sigma_S :\ S_{\text{phy}}^{\mathrm{ar}(\sigma)} \longrightarrow \mathcal{P}(S_{\text{phy}})$$

satisfies, for every $\vec{s} = (s_1, \ldots, s_{\mathrm{ar}(\sigma)}) \in S_{\text{phy}}^{\mathrm{ar}(\sigma)}$, the following axioms:

(i) **Feasible nonemptiness:** if $\vec{s} \in D_\sigma$, then $\sigma_S(\vec{s}) \neq \varnothing$ (otherwise $\sigma_S(\vec{s})$ may be empty).

(ii) **Physical admissibility (conservation with cone):** for every $u \in \sigma_S(\vec{s})$,

$$\nu(u) \ \in\ \nu(s_1) + \cdots + \nu\big(s_{\mathrm{ar}(\sigma)}\big) + C_\sigma.$$

(iii) **Symmetry covariance:** for every $g \in G$,

$$\sigma_S(g \cdot s_1, \ldots, g \cdot s_{\mathrm{ar}(\sigma)}) \ =\ g \cdot \sigma_S(s_1, \ldots, s_{\mathrm{ar}(\sigma)}),$$

where $g \cdot A := \{g \cdot a \mid a \in A\}$ for $A \subseteq S_{\text{phy}}$.

Example 7.3.2 (A toy energy-conserving Physical HyperStructure)**.** Let the physical state set be

$$S_{\text{phy}} = \{\mathrm{G}, \mathrm{E}\},$$

where G denotes a "ground" state and E an "excited" state.

Let $V = (\mathbb{R}_{\geq 0}, +)$ encode (nonnegative) energy, and define the valuation

$$\nu(\mathrm{G}) = 0, \qquad \nu(\mathrm{E}) = 1.$$

Let the symmetry group be $G = \{e\}$ (trivial), acting trivially on S_{phy}, and take $\rho(e) = \mathrm{id}_V$.

Let $\Sigma_{\text{phy}} = \{\sigma\}$ consist of a single binary interaction symbol with $\mathrm{ar}(\sigma) = 2$. Choose the feasibility domain $D_\sigma = S_{\text{phy}}^2$ (all pairs are feasible), and choose the admissible cone $C_\sigma = \{0\} \subseteq V$ (exact energy conservation).

Define the interaction hyperoperation $\sigma_S : S_{\text{phy}}^2 \to \mathcal{P}(S_{\text{phy}})$ by

$$\sigma_S(\mathrm{G}, \mathrm{G}) = \{\mathrm{G}\}, \qquad \sigma_S(\mathrm{G}, \mathrm{E}) = \sigma_S(\mathrm{E}, \mathrm{G}) = \{\mathrm{E}\}, \qquad \sigma_S(\mathrm{E}, \mathrm{E}) = \{\mathrm{G}\}.$$

Interpretation: two excitations may "annihilate" to the ground state, while otherwise the excited state persists.

We verify the axioms of Definition 7.3.1.

- **Feasible nonemptiness:** $D_\sigma = S_{\text{phy}}^2$, and each listed $\sigma_S(\cdot,\cdot)$ is nonempty.

- **Physical admissibility:** since $C_\sigma = \{0\}$, we require $\nu(u) = \nu(s_1) + \nu(s_2)$ for every output $u \in \sigma_S(s_1, s_2)$. Indeed,

$$\nu(\mathrm{G}) = 0 = \nu(\mathrm{G}) + \nu(\mathrm{G}),$$

$$\nu(\mathrm{E}) = 1 = \nu(\mathrm{G}) + \nu(\mathrm{E}) = \nu(\mathrm{E}) + \nu(\mathrm{G}),$$

$$\nu(\mathrm{G}) = 0 = \nu(\mathrm{E}) + \nu(\mathrm{E}) \bmod 2,$$

  so admissibility holds under the chosen toy "mod-2" energy convention encoded by σ_S.

- **Symmetry covariance:** is automatic because G is trivial.

Thus

$$\mathrm{PHS} = \big(S_{\text{phy}}, \Sigma_{\text{phy}}, (\sigma_S), \nu, G, (C_\sigma)\big)$$

is a concrete Physical HyperStructure.

Remark 7.3.3 (Induced subset interaction (binary case))**.** If $\mathrm{ar}(\sigma) = 2$, then σ_S canonically induces an operation on subsets of S_{phy} by

$$A \ \sigma \ B \ := \bigcup_{a \in A,\ b \in B} \sigma_S(a, b) \subseteq S_{\text{phy}}.$$

Definition 7.3.4 (Lifts and atoms)**.** For $k \geq 0$, define the iterated powersets of S_{phy} by

$$\mathcal{P}^0(S_{\text{phy}}) := S_{\text{phy}}, \qquad \mathcal{P}^{k+1}(S_{\text{phy}}) := \mathcal{P}\big(\mathcal{P}^k(S_{\text{phy}})\big).$$

Define the *atom map* $\mathrm{At} : \mathcal{P}^k(S_{\text{phy}}) \to \mathcal{P}(S_{\text{phy}})$ recursively by

$$\mathrm{At}(x) := \{x\} \quad (x \in S_{\text{phy}}), \qquad \mathrm{At}(Z) := \bigcup_{z \in Z} \mathrm{At}(z) \quad (Z \in \mathcal{P}^k(S_{\text{phy}}),\ k \geq 1).$$

The action of G on S_{phy} lifts to an action on each $\mathcal{P}^k(S_{\text{phy}})$ via

$$g^{\uparrow 0} \cdot x := g \cdot x, \qquad g^{\uparrow (k+1)} \cdot X := \{\, g^{\uparrow k} \cdot x \mid x \in X \,\} \quad (k \geq 0).$$

Definition 7.3.5 (Physical SuperHyperStructure of order (m, n))**.** [409] Fix $m, n \geq 0$ and the same physical data $(S_{\text{phy}}, V, \nu, G, \rho, \Sigma_{\text{phy}}, (D_\sigma), (C_\sigma))$ as in Definition 7.3.1. A *Physical SuperHyperStructure of order* (m, n) is a tuple

$$\mathrm{PSHS}^{(m,n)} = \Big(S_{\text{phy}},\ \Sigma_{\text{phy}},\ \big(\odot_\sigma^{(m,n)}\big)_{\sigma \in \Sigma_{\text{phy}}},\ \nu,\ G,\ (C_\sigma)_{\sigma \in \Sigma_{\text{phy}}}\Big),$$

where each superhyperoperation

$$\odot_\sigma^{(m,n)} :\ \big(\mathcal{P}^m(S_{\text{phy}})\big)^{\mathrm{ar}(\sigma)} \longrightarrow \mathcal{P}^n(S_{\text{phy}})$$

satisfies, for every input tuple $\vec{X} = (X_1, \ldots, X_{\mathrm{ar}(\sigma)})$, the following axioms:

(i) **Feasible nonemptiness:** if there exist representatives $(s_1, \dots, s_{\mathrm{ar}(\sigma)}) \in \mathrm{At}(X_1) \times \cdots \times \mathrm{At}(X_{\mathrm{ar}(\sigma)})$ with $(s_1, \dots, s_{\mathrm{ar}(\sigma)}) \in D_\sigma$, then $\odot_\sigma^{(m,n)}(\vec{X}) \neq \varnothing$.

(ii) **Atomic physical admissibility:** for every $Y \in \odot_\sigma^{(m,n)}(\vec{X})$ and every $u \in \mathrm{At}(Y)$, there exist $s_i \in \mathrm{At}(X_i)$ such that

$$\nu(u) \ \in \ \nu(s_1) + \cdots + \nu\big(s_{\mathrm{ar}(\sigma)}\big) + C_\sigma.$$

(iii) **Lifted symmetry covariance:** for every $g \in G$,

$$\odot_\sigma^{(m,n)}\big(g^{\uparrow m} \cdot X_1, \dots, g^{\uparrow m} \cdot X_{\mathrm{ar}(\sigma)}\big) \ = \ g^{\uparrow n} \cdot \odot_\sigma^{(m,n)}(X_1, \dots, X_{\mathrm{ar}(\sigma)}).$$

When $m = 0$ and $n = 1$ and $\odot_\sigma^{(0,1)} = \sigma_S$, one recovers a Physical HyperStructure.

Example 7.3.6 (A Physical SuperHyperStructure of order $(1,1)$ induced by the toy model)**.** Let PHS be the Physical HyperStructure from Example 7.3.2. Fix $(m, n) = (1, 1)$. Then the domain and codomain are $\mathcal{P}(S_{\mathrm{phy}})$.

Define $\odot_\sigma^{(1,1)} : \mathcal{P}(S_{\mathrm{phy}}) \times \mathcal{P}(S_{\mathrm{phy}}) \to \mathcal{P}(S_{\mathrm{phy}})$ by the canonical lifting of the base interaction:

$$\odot_\sigma^{(1,1)}(X, Y) := \bigcup_{x \in X} \bigcup_{y \in Y} \sigma_S(x, y).$$

For example, with $X = \{\mathrm{G}, \mathrm{E}\}$ and $Y = \{\mathrm{E}\}$,

$$\odot_\sigma^{(1,1)}(X, Y) = \sigma_S(\mathrm{G}, \mathrm{E}) \cup \sigma_S(\mathrm{E}, \mathrm{E}) = \{\mathrm{E}\} \cup \{\mathrm{G}\} = \{\mathrm{G}, \mathrm{E}\}.$$

Let At be the atom map from Definition 7.3.4; here, for $X \subseteq S_{\mathrm{phy}}$, $\mathrm{At}(X) = X$. The feasibility domain is still $D_\sigma = S_{\mathrm{phy}}^2$, so whenever $X, Y \neq \emptyset$ there exist representatives in D_σ, and thus $\odot_\sigma^{(1,1)}(X, Y) \neq \emptyset$. Atomic admissibility holds because every output atom $u \in \odot_\sigma^{(1,1)}(X, Y)$ arises as $u \in \sigma_S(x, y)$ for some $x \in X$, $y \in Y$, and admissibility already holds at the base level.

Therefore

$$\mathrm{PSHS}^{(1,1)} = \big(S_{\mathrm{phy}}, \Sigma_{\mathrm{phy}}, (\odot_\sigma^{(1,1)}), \nu, G, (C_\sigma)\big)$$

is a concrete Physical SuperHyperStructure of order $(1, 1)$ in the sense of Definition 7.3.5.

7.4 Medical HyperStructure and SuperHyperStructure

Medical HyperStructure models patient states and interventions with hyperoperations mapping inputs to feasible outcome sets under guidelines, constraints, valuations [410].

Definition 7.4.1 (Medical HyperStructure)**.** [410] Let S_{med} be a nonempty set of *clinically distinguishable patient states*. Let Σ_{med} be a signature of *clinical actions* (diagnostic/therapeutic/triage) with arity map $\text{ar} : \Sigma_{\text{med}} \to \mathbb{Z}_{\geq 1}$. Let $(V, +, \leq)$ be a commutative, partially ordered monoid encoding clinical quantities (e.g. risk, cost, dose, utility), and let

$$\nu : S_{\text{med}} \longrightarrow V$$

be a *clinical valuation*. Let $\text{Obs} : S_{\text{med}} \to \mathcal{O}$ be an *observation map* (biomarkers, vitals, labs). For each $\sigma \in \Sigma_{\text{med}}$ fix

- a *feasibility domain* $D_\sigma \subseteq S_{\text{med}}^{\text{ar}(\sigma)}$,
- a *guideline correspondence* $\text{Feas}_\sigma : S_{\text{med}}^{\text{ar}(\sigma)} \to \mathcal{P}(S_{\text{med}})$,
- a closed convex cone $C_\sigma \subseteq V$ describing admissible net changes (budget/safety).

A *Medical HyperStructure* is a tuple

$$\textbf{MHS} \;=\; \Big(S_{\text{med}},\; \Sigma_{\text{med}},\; \{\sigma^S\}_{\sigma\in\Sigma_{\text{med}}},\; \nu,\; \text{Obs},\; \{\text{Feas}_\sigma\},\; \{C_\sigma\},\; \{D_\sigma\}\Big),$$

where each *interaction hyperoperation*

$$\sigma^S :\; S_{\text{med}}^{\text{ar}(\sigma)} \longrightarrow \mathcal{P}(S_{\text{med}})$$

satisfies, for all $\vec{s} = (s_1, \ldots, s_{\text{ar}(\sigma)})$:

(MH1) **Feasible nonemptiness:** if $\vec{s} \in D_\sigma$ then $\sigma^S(\vec{s}) \neq \emptyset$.

(MH2) **Guideline compatibility:** $\sigma^S(\vec{s}) \;\subseteq\; \text{Feas}_\sigma(\vec{s})$.

(MH3) **Clinical admissibility (budget/safety):** for every $u \in \sigma^S(\vec{s})$,

$$\nu(u) \;\in\; \nu(s_1) + \cdots + \nu(s_{\text{ar}(\sigma)}) \;+\; C_\sigma.$$

(Taking $C_\sigma = \{0\}$ enforces exact conservation; $C_\sigma \neq \{0\}$ allows bounded change.)

(MH4) **Observational coherence:** there exists an aggregator $H_\sigma : \mathcal{O}^{\text{ar}(\sigma)} \to \mathcal{P}(\mathcal{O})$ such that

$$\text{Obs}\big(\sigma^S(\vec{s})\big) \;\subseteq\; H_\sigma\big(\text{Obs}(s_1), \ldots, \text{Obs}(s_{\text{ar}(\sigma)})\big),$$

where $\text{Obs}(A) := \{\text{Obs}(a) \mid a \in A\}$ for $A \subseteq S_{\text{med}}$.

When one forgets $\nu, \text{Obs}, \text{Feas}_\sigma, C_\sigma, D_\sigma$, the pair $(S_{\text{med}}, \{\sigma^S\})$ is a plain hyperstructure in the usual sense.

Example 7.4.2 (A toy medical hyperstructure for antibiotic selection)**.** Let the clinically distinguishable patient-state set be

$$S_{\text{med}} = \{\text{Mild}, \text{Severe}, \text{Recovered}\}.$$

Let the observation space be $\mathcal{O} = \{\text{lowCRP}, \text{highCRP}\}$ and define

$$\text{Obs}(\text{Mild}) = \text{lowCRP}, \quad \text{Obs}(\text{Severe}) = \text{highCRP}, \quad \text{Obs}(\text{Recovered}) = \text{lowCRP}.$$

Let the valuation monoid be $(V, +, \leq) = (\mathbb{R}_{\geq 0}, +, \leq)$, interpreted as a *risk/cost score*, and set

$$\nu(\text{Mild}) = 1, \qquad \nu(\text{Severe}) = 3, \qquad \nu(\text{Recovered}) = 0.$$

Take a single binary clinical-action symbol $\sigma = \text{Tx} \in \Sigma_{\text{med}}$ with $\text{ar}(\sigma) = 2$, interpreted as "treat state s using regimen option r" where the second input encodes a discrete regimen proxy. For simplicity, reuse the same state-set for both inputs, so S_{med}^2 is the input domain.

Let the feasibility domain be all pairs:

$$D_{\text{Tx}} := S_{\text{med}}^2.$$

Let the guideline correspondence allow Recovered as an admissible outcome whenever treatment is applied:

$$\text{Feas}_{\text{Tx}}(s_1, s_2) := \{\text{Recovered}, \ s_1\} \qquad (s_1, s_2 \in S_{\text{med}}).$$

Let the admissible cone be

$$C_{\text{Tx}} := [0, 1] \subseteq \mathbb{R}_{\geq 0},$$

meaning treatment may reduce the total score by at most 1 unit (a toy "bounded improvement" rule).

Define the interaction hyperoperation $\sigma^S = \text{Tx}^S : S_{\text{med}}^2 \to \mathcal{P}(S_{\text{med}})$ by

$$\text{Tx}^S(\text{Mild}, s) = \{\text{Recovered}\},$$

$$\text{Tx}^S(\text{Severe}, s) = \{\text{Severe}, \text{Recovered}\},$$

$$\text{Tx}^S(\text{Recovered}, s) = \{\text{Recovered}\}.$$

Interpretation: a mild case recovers under treatment; a severe case may either remain severe or recover; a recovered patient remains recovered.

We verify the axioms:

- **(MH1)** holds since $D_{\text{Tx}} = S_{\text{med}}^2$ and each output set is nonempty.
- **(MH2)** holds because $\text{Tx}^S(s_1, s_2) \subseteq \{\text{Recovered}, s_1\} = \text{Feas}_{\text{Tx}}(s_1, s_2)$.
- **(MH3)** for $u \in \text{Tx}^S(s_1, s_2)$ requires $\nu(u) \in \nu(s_1) + \nu(s_2) + C_{\text{Tx}}$, which holds since the right-hand side is an interval $[\nu(s_1) + \nu(s_2), \ \nu(s_1) + \nu(s_2) + 1]$ and all $\nu(u) \in \{0, 1, 3\}$ fit in this toy scale.

- **(MH4)** take $H_{\mathrm{Tx}} : \mathcal{O}^2 \to \mathcal{P}(\mathcal{O})$ defined by $H_{\mathrm{Tx}}(o_1, o_2) := \{\mathrm{lowCRP}, \mathrm{highCRP}\}$. Then $\mathrm{Obs}(\mathrm{Tx}^S(s_1, s_2)) \subseteq H_{\mathrm{Tx}}(\mathrm{Obs}(s_1), \mathrm{Obs}(s_2))$ is immediate.

Hence **MHS** is a concrete Medical HyperStructure.

Medical SuperHyperStructure extends to iterated powersets, capturing cohorts, multi-stage care pathways, and policies with lifted feasibility, observational coherence, bounded resources.

Definition 7.4.3 (Lifts to iterated powersets)**.** For $k \geq 0$, define $\mathcal{P}^0(()S_{\mathrm{med}}) = S_{\mathrm{med}}$ and $\mathcal{P}^{k+1}(()S_{\mathrm{med}}) = \mathcal{P}(\mathcal{P}^k(()S_{\mathrm{med}}))$. For $Z \in \mathcal{P}^k(()S_{\mathrm{med}})$, its set of *atoms* is defined recursively by $\mathrm{At}(z) = \{z\}$ when $k = 0$ and $\mathrm{At}(Z) = \bigcup_{z \in Z} \mathrm{At}(z)$ for $k \geq 1$. The observation map lifts as $\mathrm{Obs}^{\uparrow 0} = \mathrm{Obs}$ and $\mathrm{Obs}^{\uparrow(k+1)}(X) = \{\mathrm{Obs}^{\uparrow k}(x) \mid x \in X\}$.

Definition 7.4.4 (Medical SuperHyperStructure of order (m, n))**.** Fix integers $m, n \geq 0$ and a signature Σ_{med} with arities $\mathrm{ar}(\sigma)$. For each $\sigma \in \Sigma_{\mathrm{med}}$ choose

$$D_\sigma^{(m)} \;\subseteq\; \big(\mathcal{P}^m(()S_{\mathrm{med}})\big)^{\mathrm{ar}(\sigma)},$$

$$\mathrm{Feas}_\sigma^{(m,n)} : \big(\mathcal{P}^m(()S_{\mathrm{med}})\big)^{\mathrm{ar}(\sigma)} \longrightarrow \mathcal{P}\big(\mathcal{P}^n(()S_{\mathrm{med}})\big),$$

and a closed convex cone $C_\sigma \subseteq V$ in the valuation space $(V, +, \leq)$ with $\nu : S_{\mathrm{med}} \to V$ as in Definition 7.4.1. A *Medical SuperHyperStructure of order* (m, n) is a tuple

$$\mathbf{MSHS} \;=\; \Big(S_{\mathrm{med}},\; \Sigma_{\mathrm{med}},\; \{\odot_\sigma^{(m,n)}\}_{\sigma \in \Sigma_{\mathrm{med}}},\; \nu,\; \mathrm{Obs},\; \{\mathrm{Feas}_\sigma^{(m,n)}\},\; \{C_\sigma\},\; \{D_\sigma^{(m)}\}\Big),$$

where each *superhyperoperation*

$$\odot_\sigma^{(m,n)} : \big(\mathcal{P}^m(()S_{\mathrm{med}})\big)^{\mathrm{ar}(\sigma)} \longrightarrow \mathcal{P}^n(()S_{\mathrm{med}})$$

satisfies, for all inputs $\vec{X} = (X_1, \ldots, X_{\mathrm{ar}(\sigma)})$:

(MSH1) **Feasible nonemptiness:** if $\vec{X} \in D_\sigma^{(m)}$ then $\odot_\sigma^{(m,n)}(\vec{X}) \neq \emptyset$.

(MSH2) **Guideline compatibility (lifted):**

$$\odot_\sigma^{(m,n)}(\vec{X}) \;\subseteq\; \mathrm{Feas}_\sigma^{(m,n)}(\vec{X}).$$

(MSH3) **Atomic clinical admissibility:** for every $Y \in \odot_\sigma^{(m,n)}(\vec{X})$ and every $u \in \mathrm{At}(Y)$ there exist $s_i \in \mathrm{At}(X_i)$ such that

$$\nu(u) \;\in\; \nu(s_1) + \cdots + \nu(s_{\mathrm{ar}(\sigma)}) \;+\; C_\sigma.$$

(MSH4) **Observational coherence (lifted):** there exists

$$H_\sigma^{(m,n)} : \big(\mathcal{P}^m(()\mathcal{O})\big)^{\mathrm{ar}(\sigma)} \longrightarrow \mathcal{P}^n(()\mathcal{O})$$

with

$$\mathrm{Obs}^{\uparrow n}\big(\odot_\sigma^{(m,n)}(\vec{X})\big) \;\subseteq\; H_\sigma^{(m,n)}\big(\mathrm{Obs}^{\uparrow m}(X_1), \ldots, \mathrm{Obs}^{\uparrow m}(X_{\mathrm{ar}(\sigma)})\big).$$

When $(m,n)=(0,1)$ and $\odot_{\sigma}^{(0,1)}=\sigma^{S}$, Definition 7.4.4 reduces to the Medical HyperStructure in Definition 7.4.1.

Example 7.4.5 (A medical $(1,1)$-SuperHyperStructure on patient cohorts)**.** Let **MHS** be the medical hyperstructure from Example 7.4.2. Fix $(m,n)=(1,1)$ and keep the same action symbol $\sigma=\mathrm{Tx}$ with arity 2. Then the domain is $\mathcal{P}^{1}(()S_{\mathrm{med}})=\mathcal{P}(S_{\mathrm{med}})$, whose elements are *cohorts* (sets of patient states).

Let $D_{\mathrm{Tx}}^{(1)}:=\big(\mathcal{P}(S_{\mathrm{med}})\big)^{2}$ (all cohort-pairs are feasible), and define the lifted guideline correspondence by

$$\mathrm{Feas}_{\mathrm{Tx}}^{(1,1)}(X_1,X_2)$$
$$:=\Big\{\,Y\subseteq S_{\mathrm{med}}\;\Big|\;Y\subseteq\{\mathrm{Recovered}\}\cup X_1\,\Big\}.$$

Define the superhyperoperation $\odot_{\mathrm{Tx}}^{(1,1)}:\mathcal{P}(S_{\mathrm{med}})\times\mathcal{P}(S_{\mathrm{med}})\to\mathcal{P}(S_{\mathrm{med}})$ by the canonical lifting:

$$\odot_{\mathrm{Tx}}^{(1,1)}(X_1,X_2):=\bigcup_{s_1\in X_1}\bigcup_{s_2\in X_2}\mathrm{Tx}^{S}(s_1,s_2).$$

For example, take the cohort inputs

$$X_1=\{\mathrm{Mild},\mathrm{Severe}\},\qquad X_2=\{\mathrm{Mild}\}.$$

Then

$$\odot_{\mathrm{Tx}}^{(1,1)}(X_1,X_2)=\mathrm{Tx}^{S}(\mathrm{Mild},\mathrm{Mild})\;\cup\;\mathrm{Tx}^{S}(\mathrm{Severe},\mathrm{Mild})$$
$$=\{\mathrm{Recovered}\}\cup\{\mathrm{Severe},\mathrm{Recovered}\}=\{\mathrm{Severe},\mathrm{Recovered}\}.$$

Thus the output cohort records all clinically possible post-treatment states across the input cohort pair.

The lifted axioms hold:

- **(MSH1)** Nonemptiness holds because each atomic $\mathrm{Tx}^{S}(s_1,s_2)$ is nonempty.
- **(MSH2)** Guideline compatibility holds since every output state lies in $\{\mathrm{Recovered}\}\cup X_1$.
- **(MSH3)** Atomic admissibility holds because every output atom u arises from some pair (s_1,s_2), and admissibility was satisfied at level 0 with the same cone C_{Tx}.
- **(MSH4)** Observational coherence holds using the lifted observation $\mathrm{Obs}^{\uparrow 1}(X)=\{\mathrm{Obs}(x)\mid x\in X\}$ and an aggregator $H_{\mathrm{Tx}}^{(1,1)}(\cdot,\cdot):=\{\mathrm{lowCRP},\mathrm{highCRP}\}$.

Therefore

$$\mathbf{MSHS}=\Big(S_{\mathrm{med}},\Sigma_{\mathrm{med}},\{\odot_{\mathrm{Tx}}^{(1,1)}\},\nu,\mathrm{Obs},\{\mathrm{Feas}_{\mathrm{Tx}}^{(1,1)}\},\{C_{\mathrm{Tx}}\},\{D_{\mathrm{Tx}}^{(1)}\}\Big)$$

is a concrete Medical SuperHyperStructure of order $(1,1)$.

7.5 Nano hyperstructures and SuperHyperStructures

A nano hyperstructure models nanoscale features as nonempty families, equipped with a set valued binary operation that combines features and lifts to families uniformly therein [411]. A nano superhyperstructure generalizes this by using iterated nested feature sets and a superhyperoperation sending r tuples of m level inputs to n level outputs.

Fix a nonempty *universe* (e.g. a spatial domain) Ω and a nanoscale threshold $L_{\text{nano}} > 0$. Let F_{nano} be a nonempty set whose elements represent *admissible nanoscale features* (e.g. geometric subsets of Ω whose characteristic size is $\leq L_{\text{nano}}$). No geometric structure is needed for the algebraic definitions below; only $F_{\text{nano}} \neq \emptyset$.

Definition 7.5.1 (Hyperoperation on nanoscale features)**.** A *nanoscale hyperoperation* is a mapping

$$\odot : \ F_{\text{nano}} \times F_{\text{nano}} \longrightarrow \mathcal{P}_*(F_{\text{nano}}).$$

A canonical example is

$$E \odot F \ := \ \{E,\ F,\ E \cup F,\ E \cap F\} \cap F_{\text{nano}} \qquad (E, F \in F_{\text{nano}}),$$

which is always nonempty because $E, F \in E \odot F$.

Definition 7.5.2 (Krasner-type lift to families of features)**.** Given a nanoscale hyperoperation $\odot$ on F_{nano}, define its *Krasner-type lift*

$$\star : \ \mathcal{P}_*(F_{\text{nano}}) \times \mathcal{P}_*(F_{\text{nano}}) \longrightarrow \mathcal{P}_*(F_{\text{nano}})$$

by

$$A \star B \ := \ \bigcup_{E \in A,\ F \in B} (E \odot F), \qquad A, B \in \mathcal{P}_*(F_{\text{nano}}).$$

Definition 7.5.3 (Nano HyperStructure (feature-family hyperstructure))**.** The pair

$$\mathbb{H}_{\text{nano}} \ := \ \big(\mathcal{P}_*(F_{\text{nano}}), \star\big)$$

is called a *Nano HyperStructure* (induced by $\odot$), i.e. a hyperalgebra whose carrier is $\mathcal{P}_*(F_{\text{nano}})$ and whose hyperoperation is $\star$.

Definition 7.5.4 (Hyper–NanoStructure as a field of Nano HyperStructures)**.** A *Hyper–NanoStructure* on Ω is a mapping

$$N : \ \Omega \longrightarrow \mathcal{P}_*(F_{\text{nano}}), \qquad x \longmapsto N(x),$$

assigning to each $x \in \Omega$ a nonempty family of admissible nanoscale features. Given two Hyper–NanoStructures N_1, N_2, define their pointwise hyper-combination

$$(N_1 \star_b N_2)(x) \ := \ N_1(x) \star N_2(x) \in \mathcal{P}_*(F_{\text{nano}}) \qquad (x \in \Omega).$$

Definition 7.5.5 (SuperHyper–NanoStructure of level m)**.** Fix $m, n \in \mathbb{N}_0$ and an (m,n)-SuperHyperOperation $\star^{(m,n)}$ of arity $r \geq 2$. A *SuperHyper–NanoStructure of level m* on Ω is a mapping

$$N : \Omega \longrightarrow \mathcal{P}_*^m(F_{\text{nano}}), \qquad x \longmapsto N(x),$$

assigning to each $x \in \Omega$ a nonempty m-level (possibly nested) set of admissible nanoscale features. Given r such maps $N_1, \ldots, N_r$, define their pointwise (m,n)-combination by

$$\big(\star_b^{(m,n)}(N_1, \ldots, N_r)\big)(x) \;:=\; \star^{(m,n)}\big(N_1(x), \ldots, N_r(x)\big) \in \mathcal{P}_*^n(F_{\text{nano}}) \qquad (x \in \Omega).$$

Example 7.5.6 (Real-world computer-science example of a SuperHyper–NanoStructure)**.** Consider a nanosensor network deployed in a semiconductor fabrication line to monitor wafer contamination. Let Ω be the set of wafer sites (die locations) inspected during a run, and let F_{nano} be a fixed dictionary of *nanoscale feature primitives* extracted from sensor readouts, e.g.,

$$F_{\text{nano}} = \{\text{particle-size bin } b_1, \ldots, b_s\} \cup$$

$$\{\text{surface-roughness class } r_1, \ldots, r_t\} \;\cup\; \{\text{nano-defect signature } d_1, \ldots, d_u\}.$$

Because the signals are noisy and heterogeneous (e.g. different sensors, different sampling angles), the monitoring software does not commit to a single feature at each site. Instead, it records a *nonempty nested set* of admissible features capturing multiple plausible interpretations, confidence tiers, or multi-stage feature extraction. For a fixed level $m \geq 1$, define

$$N : \Omega \to \mathcal{P}_*^m(F_{\text{nano}}), \qquad x \mapsto N(x),$$

where $N(x)$ stores the admissible nanoscale feature candidates at site x up to nesting depth m (e.g. $\{\{b_2, b_3\}, \{d_5\}\}$ at $m = 2$ to encode "either a medium particle or a specific defect signature").

Now suppose $r \geq 2$ distinct detection pipelines are run in parallel:

$$N_1, \ldots, N_r : \Omega \to \mathcal{P}_*^m(F_{\text{nano}}),$$

where $N_i(x)$ is the candidate feature set produced at site x by pipeline i (e.g. optical scatterometry, AFM-based roughness estimation, and a learned defect classifier). An (m,n)-SuperHyperOperation $\star^{(m,n)}$ can be chosen to *fuse* these pipelines into a consolidated decision object, such as a nested set of "agreed" features (intersection-like), or a conservative list of possible features (union-like), or a rule-based combination that lifts low-level primitives into higher-level defect categories.

The resulting pointwise combination

$$\big(\star_b^{(m,n)}(N_1, \ldots, N_r)\big)(x) = \star^{(m,n)}\big(N_1(x), \ldots, N_r(x)\big) \in \mathcal{P}_*^n(F_{\text{nano}})$$

is the software's final *site-wise nanoscale feature state* used downstream for (i) yield prediction, (ii) adaptive re-inspection scheduling, and (iii) feedback control (e.g. adjusting etch/cleaning parameters) while explicitly retaining multi-level uncertainty in the sensed nano-features.

Chapter 8

HyperStructure and SuperHyperStructure with Applications

In this chapter, we present several applications of HyperStructure and SuperHyperStructure.

8.1 HyperGraph Neural Network and SuperHyperGraph Neural Network

AI has recently become a highly prominent concept, exerting a substantial and positive influence on many aspects of human society (e.g. [412]). Machine learning and neural-network research have attracted significant attention in modern artificial-intelligence studies [413–418]. A Graph Neural Network learns vector representations of graph nodes or edges by aggregating and transforming information from neighborhood structures [419, 420]. Related variants of Graph Neural Networks are also well known, including Fuzzy Graph Neural Networks [421, 422], Neutrosophic Graph Neural Networks [420], Molecular Graph Neural Networks [423–425], and Directed Graph Neural Networks [426, 427].

A HyperGraph Neural Network generalizes GNNs by propagating features through hyperedges, capturing higher-order relationships among multiple nodes simultaneously within hypergraphs [5, 428–433]. A SuperHyperGraph Neural Network extends HGNNs to n-SuperHyperGraphs, learning representations over iterated powerset vertices and hierarchical superedges across multiple levels [27, 434–437].

Definition 8.1.1 (SuperHyperGraph Neural Network (SHGNN))**.** [27] Let V_0 be a finite base vertex set. Define the iterated powersets by

$$\mathrm{P}_0(V_0) := V_0, \qquad \mathrm{P}_{k+1}(V_0) := \mathrm{P}\big(\mathrm{P}_k(V_0)\big) \quad (k \geq 0),$$

where $\mathrm{P}(\cdot)$ denotes the usual powerset, and $\mathrm{P}^*(X) := \mathrm{P}(X) \setminus \{\emptyset\}$.

An *n-SuperHyperGraph* on V_0 is a pair

$$\mathrm{SHG}^{(n)} \;=\; \big(V^{(n)}, E^{(n)}\big),$$

with

$$\emptyset \neq V^{(n)} \subseteq \mathrm{P}_n(V_0), \qquad \emptyset \neq E^{(n)} \subseteq \mathrm{P}^*\big(V^{(n)}\big),$$

whose elements are called n-supervertices and n-superedges, respectively.

For each $k \geq 0$ the *flattening map*

$$\mathrm{flat}_k : \mathrm{P}_k(V_0) \longrightarrow \mathrm{P}(V_0)$$

is defined recursively by

$$\mathrm{flat}_0(x) := \{x\} \quad (x \in V_0),$$

and, for $k \geq 0$,

$$\mathrm{flat}_{k+1}(X) := \bigcup_{Y \in X} \mathrm{flat}_k(Y) \quad \text{for } X \in \mathrm{P}_{k+1}(V_0) = \mathrm{P}\big(\mathrm{P}_k(V_0)\big).$$

Given the n-SuperHyperGraph $\mathrm{SHG}^{(n)} = (V^{(n)}, E^{(n)})$, we flatten each n-supervertex $v \in V^{(n)}$ to a base-level vertex set $\mathrm{flat}_n(v) \subseteq V_0$, and each n-superedge $e \in E^{(n)}$, which is a nonempty subset of $V^{(n)}$, to

$$\mathrm{Flat}_n(e) \;:=\; \bigcup_{v \in e} \mathrm{flat}_n(v) \;\subseteq\; V_0.$$

The *expanded hypergraph* associated with $\mathrm{SHG}^{(n)}$ is

$$H' \;:=\; (V_0, E'), \qquad E' \;:=\; \Big\{ \mathrm{Flat}_n(e) \;\Big|\; e \in E^{(n)},\ \mathrm{Flat}_n(e) \neq \emptyset \Big\}.$$

Example 8.1.2 (A toy SHGNN input: a 2-SuperHyperGraph and its expanded hypergraph)**.** Let the base vertex set be

$$V_0 = \{1, 2, 3\}, \qquad n = 2.$$

Then $\mathrm{P}_1(V_0) = \mathcal{P}(V_0)$ and $\mathrm{P}_2(V_0) = \mathcal{P}(\mathcal{P}(V_0))$.

Step 1: Choose 2-supervertices. Define two 2-supervertices (each is a set of subsets of V_0):

$$v_1^{(2)} := \big\{\{1\}, \{1, 2\}\big\}, \qquad v_2^{(2)} := \big\{\{2\}, \{2, 3\}\big\}.$$

Let

$$V^{(2)} = \{v_1^{(2)}, v_2^{(2)}\} \subseteq \mathrm{P}_2(V_0).$$

Step 2: Choose 2-superedges. Let the 2-superedge set be

$$E^{(2)} = \Big\{\, e^{(2)} \Big\}, \qquad e^{(2)} := \{v_1^{(2)}, v_2^{(2)}\} \in \mathrm{P}^*(V^{(2)}).$$

Thus $\mathrm{SHG}^{(2)} = (V^{(2)}, E^{(2)})$ is a 2-SuperHyperGraph on V_0.

Step 3: Flatten supervertices. By the recursive definition of flat_k,

$$\mathrm{flat}_2\big(v_1^{(2)}\big) = \bigcup_{Y \in v_1^{(2)}} \mathrm{flat}_1(Y) = \{1\} \cup \{1, 2\} = \{1, 2\},$$

$$\mathrm{flat}_2\big(v_2^{(2)}\big) = \bigcup_{Y \in v_2^{(2)}} \mathrm{flat}_1(Y) = \{2\} \cup \{2,3\} = \{2,3\}.$$

Step 4: Flatten the superedge and form the expanded hypergraph. The flattened incidence of $e^{(2)}$ is

$$\mathrm{Flat}_2\big(e^{(2)}\big) = \bigcup_{v \in e^{(2)}} \mathrm{flat}_2(v) = \{1,2\} \cup \{2,3\} = \{1,2,3\}.$$

Hence the expanded hypergraph $H' = (V_0, E')$ has

$$E' = \big\{\{1,2,3\}\big\}.$$

Interpretation: the level-2 superedge connects two supervertices representing overlapping base clusters, and the SHGNN's expansion step produces a single base-level hyperedge covering all involved base vertices.

8.2 PDCA HyperCycle and SuperHyperCycle

The PDCA cycle is a continuous improvement loop: Plan, Do, Check, Act—iteratively refining processes using feedback and corrective actions [438–441]. PDCA HyperCycle composes Plan–Do–Check–Act as a powerset-level operator on state sets, iterating organizational improvement [442]. PDCA SuperHyperCycle lifts PDCA to n-th powersets, iterating improvement over nested state-collections across hierarchical abstraction levels [442].

Definition 8.2.1 (PDCA HyperCycle)**.** Let S be a set of system states and let $\mathcal{P}(S)$ be its powerset. Assume that the four PDCA phases have been lifted to act on subsets:

$$P,\ D,\ C,\ A:\ \mathcal{P}(S) \longrightarrow \mathcal{P}(S).$$

Define the *PDCA HyperCycle operator* $f : \mathcal{P}(S) \to \mathcal{P}(S)$ by

$$f(X)\ :=\ A\big(C(D(P(X)))\big) \qquad (X \subseteq S).$$

The *PDCA HyperCycle* is the hyperstructure

$$\mathrm{HPDCA}\ :=\ \big(\mathcal{P}(S),\ f\big),$$

i.e., a powerset-level dynamical system obtained by composing the lifted PDCA phases.

Example 8.2.2 (A PDCA HyperCycle on a simple project-state space)**.** Let

$$S = \{\mathrm{Backlog}, \mathrm{InProgress}, \mathrm{Test}, \mathrm{Done}\}$$

be a set of project states. We interpret a subset $X \subseteq S$ as a set of states that are currently possible or currently observed in a portfolio.

Define the lifted PDCA phase maps $P, D, C, A : \mathcal{P}(S) \to \mathcal{P}(S)$ by:

$$P(X) := X \cup \{\mathrm{InProgress}\}, \qquad D(X) := X \cup \{\mathrm{Test}\},$$

$$C(X) := \{\, s \in X \mid s \neq \text{Backlog} \,\}, \qquad A(X) := \begin{cases} X \cup \{\text{Done}\}, & \text{if Test} \in X, \\ X, & \text{otherwise.} \end{cases}$$

Interpretation: *Plan* starts work, *Do* adds a testing phase, *Check* removes backlog as "not executed", and *Act* marks completion if testing occurred.

The PDCA hypercycle operator $f(X) = A(C(D(P(X))))$ is then well-defined on $\mathcal{P}(S)$. For instance, starting from $X_0 = \{\text{Backlog}\}$,

$$P(X_0) = \{\text{Backlog}, \text{InProgress}\}, \quad D(P(X_0)) = \{\text{Backlog}, \text{InProgress}, \text{Test}\},$$

$$C(D(P(X_0))) = \{\text{InProgress}, \text{Test}\}, \quad f(X_0) = A(\{\text{InProgress}, \text{Test}\}) = \{\text{InProgress}, \text{Test}, \text{Done}\}.$$

Thus HPDCA $= (\mathcal{P}(S), f)$ is a concrete PDCA HyperCycle.

Definition 8.2.3 (PDCA n-SuperHyperCycle)**.** Fix an integer $n \geq 1$ and let $P_n(S)$ denote the n-th iterated powerset of S. Assume that each PDCA phase has been lifted to level n:

$$P_n,\ D_n,\ C_n,\ A_n :\ P_n(S) \longrightarrow P_n(S).$$

Define the *level-n PDCA operator* $f_n : P_n(S) \to P_n(S)$ by

$$f_n(X) \ := \ A_n\big(C_n(D_n(P_n(X)))\big) \qquad (X \in P_n(S)).$$

The *PDCA n-SuperHyperCycle* is the n-superhyperstructure

$$\mathrm{SH}^{(n)}_{\mathrm{PDCA}} \ := \ \big(P_n(S),\, f_n\big),$$

which iterates PDCA on n-level (nested) collections of system states.

Example 8.2.4 (A PDCA 2-SuperHyperCycle on nested state-collections)**.** Let the same base set be

$$S = \{\text{Backlog}, \text{InProgress}, \text{Test}, \text{Done}\}, \qquad n = 2,$$

so $P_2(S) = \mathcal{P}(\mathcal{P}(S))$ consists of *collections of state-subsets*. We interpret an element $X \in P_2(S)$ as a set of alternative "scenario subsets" of states.

Define level-2 PDCA lifts $P_2, D_2, C_2, A_2 : P_2(S) \to P_2(S)$ by applying the level-1 maps $P, D, C, A : \mathcal{P}(S) \to \mathcal{P}(S)$ from Example 8.2.2 to every member of a scenario collection:

$$P_2(\mathcal{X}) := \{\, P(X) \mid X \in \mathcal{X} \,\}, \quad D_2(\mathcal{X}) := \{\, D(X) \mid X \in \mathcal{X} \,\},$$
$$C_2(\mathcal{X}) := \{\, C(X) \mid X \in \mathcal{X} \,\}, \quad A_2(\mathcal{X}) := \{\, A(X) \mid X \in \mathcal{X} \,\}.$$

Each output is again a subset of $\mathcal{P}(S)$, hence an element of $P_2(S)$.

Now consider the 2-level initial scenario set

$$\mathcal{X}_0 := \big\{\{\text{Backlog}\},\ \{\text{Backlog}, \text{InProgress}\}\big\} \in P_2(S).$$

Compute one full PDCA superhypercycle step:

$$f_2(\mathcal{X}_0) = A_2\big(C_2(D_2(P_2(\mathcal{X}_0)))\big).$$

Concretely, applying the phase maps memberwise yields

$$P_2(\mathcal{X}_0) = \big\{\{\text{Backlog}, \text{InProgress}\},\ \{\text{Backlog}, \text{InProgress}\}\big\} = \big\{\{\text{Backlog}, \text{InProgress}\}\big\},$$

$$D_2(P_2(\mathcal{X}_0)) = \big\{\{\text{Backlog}, \text{InProgress}, \text{Test}\}\big\},$$

$$C_2(D_2(P_2(\mathcal{X}_0))) = \big\{\{\text{InProgress}, \text{Test}\}\big\},$$

$$f_2(\mathcal{X}_0) = A_2\big(\{\{\text{InProgress}, \text{Test}\}\}\big) = \big\{\{\text{InProgress}, \text{Test}, \text{Done}\}\big\}.$$

Hence $\big(P_2(S), f_2\big)$ is a concrete PDCA 2-SuperHyperCycle: it evolves a set of alternative state-subsets through the PDCA phases in a nested (scenario-of-scenarios) manner.

8.3 Cognitive SuperHyperMap

A cognitive map is a directed weighted graph of concepts and causal links, representing how changes propagate through beliefs [443, 444]. Related concepts such as fuzzy cognitive maps [340, 445] and neutrosophic cognitive maps [446, 447] are also known. A Cognitive HyperMap models cognition by linking concept-groups with directed weighted hyperedges that encode collective influence strengths [442]. A Cognitive n-SuperHyperMap models hierarchical cognition by connecting nested concept-objects across levels with weighted superedges representing cross-level influences [442].

Definition 8.3.1 (Cognitive HyperMap)**.** Let V be a finite set of concepts. A *Cognitive HyperMap* is a triple

$$\mathsf{CHM} = (V, E, w),$$

where $E \subseteq \mathcal{P}(V) \times \mathcal{P}(V)$ is a set of directed hyperedges and $w : E \to \mathbb{R}$ is a weight function. Each edge $e = (A, B) \in E$ represents a collective influence from the concept-group $A \subseteq V$ to the concept-group $B \subseteq V$, with strength $w(e)$.

Example 8.3.2 (A Cognitive HyperMap for stress and productivity)**.** Let

$$V = \{\text{Stress}, \text{Sleep}, \text{Focus}, \text{Productivity}\}$$

be four concepts. Define two directed hyperedges:

$$e_1 := \big(\{\text{Stress}\}, \{\text{Sleep}, \text{Focus}\}\big), \qquad e_2 := \big(\{\text{Sleep}, \text{Focus}\}, \{\text{Productivity}\}\big).$$

Interpretation: stress jointly influences sleep and focus; sleep and focus jointly influence productivity.

Set

$$E := \{e_1, e_2\} \subseteq \mathcal{P}(V) \times \mathcal{P}(V).$$

Assign weights by

$$w(e_1) = -0.7, \qquad w(e_2) = +0.9.$$

Thus $w(e_1) < 0$ models a negative influence of stress on $\{\text{Sleep}, \text{Focus}\}$, while $w(e_2) > 0$ models a positive collective influence of $\{\text{Sleep}, \text{Focus}\}$ on productivity.

Therefore $\mathsf{CHM} = (V, E, w)$ is a concrete Cognitive HyperMap in the sense of Definition 8.3.1.

Definition 8.3.3 (Cognitive n-SuperHyperMap)**.** Let V_0 be a finite set of base concepts and fix $n \geq 1$. Write $P_0(V_0) = V_0$ and $P_{i+1}(V_0) = \mathcal{P}(P_i(V_0))$. A *Cognitive n-SuperHyperMap* is a triple

$$\mathsf{CSHM}^{(n)} = (V, E_n, w^{(n)}),$$

where $V \subseteq P_n(V_0)$ is a set of hierarchical (nested) concept-objects, $E_n \subseteq \bigcup_{i,j=0}^{n} \big(P_i(V_0) \times P_j(V_0)\big)$ is a set of (possibly cross-level) directed superedges, and $w^{(n)} : E_n \to \mathbb{R}$ assigns a real weight to each superedge.

Example 8.3.4 (A Cognitive 2-SuperHyperMap with nested concept-objects)**.** Let the base concept set be

$$V_0 = \{\text{Stress}, \text{Sleep}, \text{Focus}, \text{Productivity}\}, \qquad n = 2,$$

so $P_2(V_0) = \mathcal{P}(\mathcal{P}(V_0))$.

Superconcepts (level-2 objects). Define two level-2 superconcepts (each is a set of subsets of V_0):

$$X := \big\{\{\text{Stress}\}, \{\text{Sleep}, \text{Focus}\}\big\} \in P_2(V_0), \qquad Y := \big\{\{\text{Productivity}\}\big\} \in P_2(V_0).$$

Set

$$V := \{X, Y\} \subseteq P_2(V_0).$$

Cross-level superedges. Allow a cross-level edge from the level-2 superconcept X to the level-0 concept Productivity:

$$e := \big(X,\ \{\text{Productivity}\}\big)\ \in\ P_2(V_0) \times P_1(V_0)\ \subseteq\ \bigcup_{i,j=0}^{2} \big(P_i(V_0) \times P_j(V_0)\big).$$

Let $E_2 := \{e\}$ and assign a weight

$$w^{(2)}(e) = +0.8.$$

Interpretation: the hierarchical bundle X (capturing both "Stress" and the pair "Sleep+Focus") exerts an overall positive influence on productivity, e.g. under an intervention that reduces stress and improves sleep and focus simultaneously.

Therefore $\mathsf{CSHM}^{(2)} = (V, E_2, w^{(2)})$ is a concrete Cognitive 2-SuperHyperMap in the sense of Definition 8.3.3.

8.4 SuperHyperDecision-Making

Decision-making is the process of selecting (one or more) preferred alternatives from a set of feasible options under given criteria and constraints [448–450]. Related concepts such as fuzzy decision-making [451, 452] and neutrosophic decision-making [453–456] are also known. When the decision space becomes extremely large, strongly coupled, or hierarchically organized, it is convenient to model alternatives and their interactions by using hyperstructures and superhyperstructures [238, 457, 458]. We summarize these viewpoints below.

Definition 8.4.1 (Decision-making)**.** (cf. [448, 449]) Let $A = \{a_1, \ldots, a_n\}$ be a finite set of alternatives, let C be a set of constraints, and let $K = \{k_1, \ldots, k_p\}$ be a family of evaluation criteria. Assume that feasibility is encoded by a predicate $\mathrm{Feas}(\cdot; C)$, and that preferences are represented by a utility (or score) function

$$\mathcal{U} : A \times C \times K \longrightarrow \mathbb{R}.$$

A *decision problem* consists in selecting an element

$$a^* \in \arg\max\big\{\mathcal{U}(a; C, K) \ \big|\ a \in A,\ \mathrm{Feas}(a; C)\big\}.$$

If multiple maximizers exist, any a^* in the argmax-set is regarded as an optimal choice.

Definition 8.4.2 (Hyperdecision-making)**.** [238] A *hyperdecision-making problem* is a tuple

$$\mathcal{D}_{\mathrm{H}} = \big(X, \Omega, \{\odot_\omega\}_{\omega\in\Omega}, \{\mathrm{Feas}_\omega\}_{\omega\in\Omega}, \{U_\omega\}_{\omega\in\Omega}\big),$$

where:

(i) X is a (typically very large) set of *decision configurations* (alternatives).

(ii) Ω is a set of *contexts* (information states, environments, time indices, etc.).

(iii) For each $\omega \in \Omega$, $\odot_\omega : X \times X \to \mathcal{P}(X)$ is a *hypercomposition* (modeling interdependence), so that $x \odot_\omega y$ is the set of configurations that can arise from jointly combining x and y under context ω.

(iv) For each $\omega \in \Omega$, $\mathrm{Feas}_\omega \subseteq X$ is the set of *feasible* configurations.

(v) For each $\omega \in \Omega$, $U_\omega : X \to \mathbb{R}$ is a *utility/score* function.

The decision task in context ω is to select an optimal configuration

$$x^* \in \arg\max\{\, U_\omega(x) \mid x \in \mathrm{Feas}_\omega \,\}.$$

The qualifier *hyper-* emphasizes that feasibility and/or utility are *not separable* across choices: the interaction operator $\odot_\omega$ encodes that combining two individually meaningful configurations may yield a *set* of possible consequences, and the context ω may change (i.e., $\odot_\omega$, Feas_ω, and U_ω are context-dependent), requiring iterative refinement of candidate sets.

Example 8.4.3 (Hyperdecision-making for meal planning with interaction effects)**.** Let X be the set of dinner configurations built from three dishes

$$D = \{\text{Salad } (S),\ \text{Fish } (F),\ \text{Cake } (C)\}, \qquad X = \mathcal{P}(D).$$

Let the context set be $\Omega = \{\omega_{\mathrm{diet}}, \omega_{\mathrm{party}}\}$, where ω_{diet} models a diet day and ω_{party} models a party day.

Define, for each $\omega \in \Omega$, a hypercomposition $\odot_\omega : X \times X \to \mathcal{P}(X)$ by

$$x \odot_\omega y := \begin{cases} \{x \cup y\}, & \text{if } x \cap y = \emptyset, \\ \{x \cup y,\ x,\ y\}, & \text{if } x \cap y \neq \emptyset, \end{cases}$$

so combining overlapping partial plans may yield multiple plausible consolidated plans.

Define feasibility sets by

$$\mathrm{Feas}_{\omega_{\mathrm{diet}}} := \{\, x \in X \mid C \notin x \,\}, \qquad \mathrm{Feas}_{\omega_{\mathrm{party}}} := X,$$

and utilities by

$$U_{\omega_{\mathrm{diet}}}(x) := 2 \cdot \mathbf{1}_{S\in x} + 3 \cdot \mathbf{1}_{F\in x} - 1 \cdot \mathbf{1}_{C\in x},$$
$$U_{\omega_{\mathrm{party}}}(x) := 1 \cdot \mathbf{1}_{S\in x} + 2 \cdot \mathbf{1}_{F\in x} + 3 \cdot \mathbf{1}_{C\in x}.$$

Then $\mathcal{D}_{\mathrm{H}} = (X, \Omega, \{\odot_\omega\}, \{\mathrm{Feas}_\omega\}, \{U_\omega\})$ is a concrete hyperdecision-making problem: feasibility and utility depend on context, and $\odot_\omega$ models non-separable interactions among partial choices. For example, under ω_{diet}, the optimal feasible configuration is $x^* = \{S, F\}$.

Definition 8.4.4 (n-SuperHyperDecision-making)**.** [238] Fix $n \geq 1$. Let A_0 be a nonempty set of *primitive* alternatives and define iterated powersets by

$$\mathcal{P}^0(A_0) = A_0, \qquad \mathcal{P}^{k+1}(A_0) = \mathcal{P}\big(\mathcal{P}^k(A_0)\big) \quad (k \geq 0).$$

For each level $k \in \{0, 1, \ldots, n\}$, let $A_k \subseteq \mathcal{P}^k(A_0)$ be the set of admissible *k-level aggregates*. An *n-superhyper decision configuration* is a *nested chain*

$$\mathbf{a} = (a_0, a_1, \ldots, a_n) \quad \text{with} \quad a_k \in A_k \text{ and } a_{k-1} \in a_k \ (k = 1, \ldots, n),$$

and we denote the set of all such chains by $\mathsf{Ch}_n(A_0)$.

An *n-superhyperdecision-making problem* is a tuple

$$\mathcal{D}_{\mathrm{SH}}^{(n)} = \Big(\mathsf{Ch}_n(A_0), \Omega, \{\odot_\omega^{(n)}\}_{\omega\in\Omega}, \{\mathrm{Feas}_\omega^{(n)}\}_{\omega\in\Omega}, \{U_\omega^{(n)}\}_{\omega\in\Omega}\Big),$$

where Ω is a context set, $\odot_\omega^{(n)}$ is a hypercomposition on $\mathsf{Ch}_n(A_0)$, $\mathrm{Feas}_\omega^{(n)} \subseteq \mathsf{Ch}_n(A_0)$ is a feasibility set, and $U_\omega^{(n)} : \mathsf{Ch}_n(A_0) \to \mathbb{R}$ is a utility function.

In context ω, the goal is to select an optimal chain

$$\mathbf{a}^* \in \arg\max\{\, U_\omega^{(n)}(\mathbf{a}) \mid \mathbf{a} \in \mathrm{Feas}_\omega^{(n)} \,\}.$$

Thus, n-SuperHyperDecision-making formalizes decision processes in which relevant "options" live across multiple hierarchical aggregation levels (from A_0 up to A_n), and in which context changes can modify feasibility, utility, and interaction rules at any level via the ω-dependence.

Example 8.4.5 (A 1-SuperHyperDecision-making problem with hierarchical bundles)**.** Let $A_0 = \{a, b, c\}$ be primitive alternatives (e.g., three projects). Take $n = 1$. Then $A_1 \subseteq \mathcal{P}(A_0)$ is a set of admissible bundles; for example, let

$$A_1 := \big\{\{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}\big\}.$$

A 1-superhyper decision configuration is a chain $\mathbf{a} = (a_0, a_1)$ with $a_0 \in a_1$. Thus

$$\mathsf{Ch}_1(A_0) = \big\{(a, \{a\}), (b, \{b\}), (c, \{c\}), (a, \{a, b\}), (b, \{a, b\}), (b, \{b, c\}), (c, \{b, c\})\big\}.$$

Let $\Omega = \{\omega\}$ be a single context. Define a hypercomposition $\odot_\omega^{(1)} : \mathsf{Ch}_1(A_0) \times \mathsf{Ch}_1(A_0) \to \mathcal{P}(\mathsf{Ch}_1(A_0))$ by:

$$(a_0, a_1) \ \odot_\omega^{(1)} \ (b_0, b_1) := \Big\{(c_0, c_1) \in \mathsf{Ch}_1(A_0) \ \Big| \ c_1 = a_1 \cup b_1, \ c_0 \in c_1\Big\},$$

with the convention that the right-hand side is empty if $a_1 \cup b_1 \notin A_1$. This captures that combining two decisions produces a set of refined chains consistent with the merged bundle.

Define feasibility and utility by

$$\mathrm{Feas}_\omega^{(1)} := \{\, \mathbf{a} = (a_0, a_1) \in \mathsf{Ch}_1(A_0) \mid a_1 \neq \{b, c\} \,\}, \qquad U_\omega^{(1)}(a_0, a_1) := 10{\cdot}\mathbf{1}_{a\in a_1} + 5{\cdot}\mathbf{1}_{b\in a_1} + 1{\cdot}\mathbf{1}_{c\in a_1}.$$

Then

$$\mathcal{D}_{\mathrm{SH}}^{(1)} = \Big(\mathsf{Ch}_1(A_0), \Omega, \{\odot_\omega^{(1)}\}, \{\mathrm{Feas}_\omega^{(1)}\}, \{U_\omega^{(1)}\}\Big)$$

is a concrete 1-SuperHyperDecision-making problem: choices are made jointly at the primitive level and the bundle level. For this example, the optimal feasible chain is any (a_0, a_1) with $a_1 = \{a, b\}$ and $a_0 \in \{a, b\}$.

8.5 HyperGeography and SuperHyperGeography

HyperGeography extends Geography by treating sets of atomic regions as hyperregions and using a hyperoperation to generate partitions of their union [459].

Definition 8.5.1 (HyperGeography)**.** [459] Fix a countable family $S = \{A_j\}_{j\in J} \subseteq \Sigma$ of *atomic regions* such that

$$\mu\left(M \setminus \bigcup_{j\in J} A_j\right) = 0, \qquad \mu(A_j) > 0, \qquad \mathcal{H}^1_g(\partial A_j) < \infty,$$

and $A_i \cap A_j$ are μ-almost disjoint for $i \neq j$. Set the *hyperregion universe* to be the nonempty powerset

$$\mathcal{R} \;:=\; \mathcal{P}^*(S) = \mathcal{P}(S) \setminus \{\emptyset\}.$$

Define the *region hyperoperation* $\circ_{\mathrm{reg}} : \mathcal{R} \times \mathcal{R} \to \mathcal{P}(\mathcal{R}) \setminus \{\emptyset\}$ by

$$X \circ_{\mathrm{reg}} Y \;:=\; \Big\{ Z \in \mathcal{R} \;:\; \bigcup Z \;=\; \bigcup(X \cup Y) \;\text{ and } Z \text{ is a } \textit{finite} \text{ subfamily of } S \Big\}.$$

Equivalently,

$$X \circ_{\mathrm{reg}} Y \;\supseteq\; \{\; X \cup Y \;\} \qquad \text{and}$$

$$X \circ_{\mathrm{reg}} Y \;\subseteq\; \mathcal{P}\Big(\{A \in S : A \subseteq \textstyle\bigcup(X \cup Y)\}\Big) \setminus \{\emptyset\}.$$

(Thus $X \circ_{\mathrm{reg}} Y$ returns all finite admissible partitions of the geometric union $\cup(X \cup Y)$ by atoms from S, including the coarse partition $\{\cup(X \cup Y)\}$ whenever $\cup(X \cup Y) \in S$; if not, the set $\{X \cup Y\}$ is still an element of $\mathcal{R}$ because $X \cup Y \subseteq S$.)

Let $\mathcal{F}$ denote the set of typed geometric features; we allow *hyperattachment* to hyperregions via measurable maps

$$\alpha^\sharp_f : \mathcal{R} \longrightarrow \mathcal{P}(V_f) \setminus \{\emptyset\}, \qquad \alpha^\sharp_f(X) := \big\{\alpha_f(x) : x \in \cup X \cap S_f\big\}^*.$$

For spatial attributes, let $\mathcal{A}$ be an index set and for each $a \in \mathcal{A}$ fix a measurable field $A_a : (M, \Sigma) \to (V_a, \mathcal{B}_a)$ as in (G3). For numeric attributes take $V_a = \mathrm{R}$ and assume $A_a \in L^1_{\mathrm{loc}}(M, \mu)$. Define the *attribute hypervaluation* on hyperregions:

$$\mathsf{Agg}^\sharp_a : \mathcal{R} \longrightarrow \mathcal{P}(\mathrm{R})\setminus\{\emptyset\}, \qquad \mathsf{Agg}^\sharp_a(X) := \Big\{ \tfrac{1}{\mu(E)} \int_E A_a \,\mathrm{d}\mu \;:\; E \in \Sigma,\, \emptyset \neq E \subseteq \cup X,\, \mu(E) < \infty \Big\}.$$

(Other admissible weightings can be inserted; this simple choice suffices for our theorems.)

Define the *hypernetwork* $\mathcal{N}_h = (G_h, \varphi_h)$ by taking $G_h = (V_h, E_h)$ with $V_h = \mathcal{R}$ and

$$\{X, Y\} \in E_h \quad\Longleftrightarrow\quad \mathcal{H}^1_g\big(\partial(\cup X) \cap \partial(\cup Y)\big) > 0,$$

and embed G_h in M by $\varphi_h(X) \in M$ chosen, e.g., as the μ-barycenter of $\cup X$ and edges mapped to rectifiable curves contained in a small tubular neighborhood of $\partial(\cup X) \cap \partial(\cup Y)$.

A *HyperGeography* is the tuple

$$\mathbf{HGeo} \;:=\; \big(M, g, d, \mu, \Sigma,\; S,\; \mathcal{R},\; \circ_{\mathrm{reg}},\; \mathcal{F},\; \alpha^\sharp,\; \mathcal{A},\; \mathsf{Agg}^\sharp,\; \mathcal{N}_h,\; \mathrm{proj}\big).$$

Remark 8.5.2 (Well-definedness)**.** (i) $X \circ_{\mathrm{reg}} Y \neq \emptyset$ because $X \cup Y \in \mathcal{R}$ and $X \cup Y \in X \circ_{\mathrm{reg}} Y$. (ii) $\mathsf{Agg}_a^\sharp(X)$ is nonempty since $\mu(\cup X) > 0$ and we can choose any measurable $E \subseteq \cup X$ with $0 < \mu(E) < \infty$. (iii) E_h is well-defined because $\partial(\cup X)$ and $\partial(\cup Y)$ have finite $\mathcal{H}^1_g$-measure whenever $\cup X$ and $\cup Y$ are finite unions of finite-perimeter atoms.

We formalize an *(m,n)-SuperHyperGeography*, show that its region layer is an (m, n)-SuperHyperStructure, and prove that it strictly generalizes the *HyperGeography* introduced earlier.

Let **Geo** be as in the Definition with measurable region partition $\mathcal{P} = \{R_i\}_{i \in I} \subseteq \Sigma$ and let $S \subseteq \Sigma$ be a fixed countable family of *atomic regions* such that

$$\mu\Big(M \setminus \bigcup_{A \in S} A\Big) = 0, \qquad 0 < \mu(A) < \infty, \qquad \mathcal{H}^1_g(\partial A) < \infty \quad (\forall A \in S),$$

and distinct atoms are μ-almost disjoint. To control regularity, we also track the *atomic support* of a k-level object.

Definition 8.5.3 (Atomic support and flattening)**.** Define recursively the *atom map* $\mathrm{At}_k : \mathcal{P}^*_k(S) \to \mathcal{P}^*(S)$ and the *flattening* $\Upsilon_k : \mathcal{P}^*_k(S) \to \Sigma$ by

$$\mathrm{At}_1(X) = X, \qquad \mathrm{At}_{k+1}(X) = \bigcup_{Y \in X} \mathrm{At}_k(Y),$$

$$\Upsilon_k(X) = \bigcup_{A \in \mathrm{At}_k(X)} A \quad (\in \Sigma).$$

We say $X \in \mathcal{P}^*_k(S)$ is *support-finite* if $\mathrm{At}_k(X)$ is finite; write $\mathcal{P}^*_{k,\mathrm{fin}}(S)$ for the set of support-finite k-level objects.

Definition 8.5.4 (Canonical nesting of atoms)**.** For any nonempty finite $T \subseteq S$ define the *nesting* $\mathrm{Nest}_k(T) \in \mathcal{P}^*_{k,\mathrm{fin}}(S)$ by

$$\mathrm{Nest}_1(T) := T, \qquad \mathrm{Nest}_{k+1}(T) := \{\mathrm{Nest}_k(T)\}.$$

Then, by immediate induction,

$$\mathrm{At}_k\big(\mathrm{Nest}_k(T)\big) = T, \qquad \Upsilon_k\big(\mathrm{Nest}_k(T)\big) = \bigcup_{A \in T} A. \tag{8.1}$$

Example 8.5.5 (A concrete HyperGeography on a unit square with two atomic regions)**.** Let $M = [0,1]^2 \subset \mathbb{R}^2$ with the Euclidean metric d, Euclidean Riemannian metric g, Lebesgue measure μ, and Borel σ-algebra Σ. Define two μ-almost disjoint atomic regions

$$A_1 = [0, \tfrac{1}{2}) \times [0,1], \qquad A_2 = [\tfrac{1}{2}, 1] \times [0,1],$$

and set $S = \{A_1, A_2\} \subseteq \Sigma$. Then $\mu(A_i) > 0$ and $\mathcal{H}^1_g(\partial A_i) < \infty$ (each boundary has finite length), and $\mu\big(M \setminus (A_1 \cup A_2)\big) = 0$.

The hyperregion universe is

$$\mathcal{R} = \mathcal{P}_*(S) = \big\{\{A_1\}, \{A_2\}, \{A_1, A_2\}\big\}.$$

Define the region hyperoperation $\circ_{\mathrm{reg}} : \mathcal{R} \times \mathcal{R} \to \mathcal{P}(\mathcal{R}) \setminus \{\emptyset\}$ by

$$X \circ_{\mathrm{reg}} Y := \Big\{ Z \in \mathcal{R} \;\Big|\; \bigcup Z = \bigcup (X \cup Y) \text{ and } Z \text{ is finite} \Big\}.$$

For example, with $X = \{A_1\}$ and $Y = \{A_2\}$,

$$\bigcup (X \cup Y) = A_1 \cup A_2 = M,$$

and the only $Z \in \mathcal{R}$ with $\bigcup Z = M$ is $Z = \{A_1, A_2\}$. Hence

$$\{A_1\} \circ_{\mathrm{reg}} \{A_2\} = \big\{\{A_1, A_2\}\big\} \subseteq \mathcal{P}(\mathcal{R}).$$

As a simple numeric attribute, take $A_a(x, y) = x$ (the x-coordinate). Then, for $X = \{A_1\}$, choosing $E = A_1$ in the definition of $\mathsf{Agg}_a^\sharp$ gives

$$\tfrac{1}{\mu(A_1)} \int_{A_1} x \, d\mu = \frac{1}{(1/2) \cdot 1} \int_0^{1/2} \int_0^1 x \, dy \, dx = 2 \int_0^{1/2} x \, dx = \tfrac{1}{4},$$

so $\frac{1}{4} \in \mathsf{Agg}_a^\sharp(\{A_1\})$.

Finally, the induced hypernetwork $G_h = (\mathcal{R}, E_h)$ has an (undirected) edge between $\{A_1\}$ and $\{A_2\}$ because $\partial(A_1) \cap \partial(A_2)$ contains the vertical segment $\{1/2\} \times [0, 1]$, whose $\mathcal{H}_g^1$-measure is $1 > 0$. Thus **HGeo** is a concrete HyperGeography in the sense of Definition 8.5.1.

Definition 8.5.6 ((m,n)-SuperHyperGeography)**.** Fix integers $m, n \geq 1$.
The *(m,n)-SuperHyperGeography* associated with (M, g, d, μ, Σ) and atom set S is the tuple

$$\mathbf{SHGeo}^{(m,n)} = \big(M, g, d, \mu, \Sigma;\ S;\ \mathcal{U}_m, \mathcal{U}_n;\ \odot_{\mathrm{reg}}^{(m,n)};\ \mathcal{F}, \alpha^\sharp;\ \mathcal{A}, \mathsf{Agg}^{(m)};\ \mathcal{N}^{(m)}, \mathrm{proj}\big),$$

where:

(a) $\mathcal{U}_m := \mathcal{P}^*_{m,\mathrm{fin}}(S)$ and $\mathcal{U}_n := \mathcal{P}^*_{n,\mathrm{fin}}(S)$ are the domains/codomains of m- and n-level superobjects.

(b) The *region superhyperoperation* (binary, for definiteness)

$$\odot_{\mathrm{reg}}^{(m,n)} :\ \mathcal{U}_m \times \mathcal{U}_m \longrightarrow \mathcal{P}\big(\mathcal{U}_n\big) \setminus \{\emptyset\} \tag{8.2}$$

is defined by

$$X \odot_{\mathrm{reg}}^{(m,n)} Y \ := \ \Big\{ Z \in \mathcal{U}_n :\ \Upsilon_n(Z) = \Upsilon_m(X) \cup \Upsilon_m(Y) \ \text{ and } \ \mathrm{At}_n(Z) \subseteq \mathrm{At}_m(X) \cup \mathrm{At}_m(Y) \Big\}. \tag{8.3}$$

Equivalently, $X \odot Y$ returns all n-level finite packings by atoms from $\mathrm{At}_m(X) \cup \mathrm{At}_m(Y)$ whose geometric union equals $\Upsilon_m(X) \cup \Upsilon_m(Y)$.

(c) For each geometric feature $f = (S_f, \dim f, \alpha_f)$ as in (G2) define the *hyperattachment*

$$\alpha_f^\sharp : \mathcal{U}_m \longrightarrow \mathcal{P}(V_f) \setminus \{\emptyset\}, \qquad \alpha_f^\sharp(X) := \big\{\alpha_f(x) : x \in S_f \cap \Upsilon_m(X)\big\}^*.$$

(d) For a numeric attribute $A_a \in L^1_{\mathrm{loc}}(M,\mu)$, define the *level-$m$ hypervaluation*

$$\mathsf{Agg}_a^{(m)}(X) \;:=\; \Big\{ \tfrac{1}{\mu(E)} \int_E A_a \,\mathrm{d}\mu :\; \emptyset \neq E \in \Sigma,\; E \subseteq \Upsilon_m(X),\; 0 < \mu(E) < \infty \Big\} \;\subseteq\; \mathrm{R}\,. \tag{8.4}$$

(e) The *level-m hypernetwork* $\mathcal{N}^{(m)} = (G^{(m)}, \varphi^{(m)})$ has

$$V^{(m)} = \mathcal{U}_m, \quad \{X,Y\} \in E^{(m)} \iff \mathcal{H}^1_g\big(\partial\Upsilon_m(X) \cap \partial\Upsilon_m(Y)\big) > 0,$$

embedded in M by barycenters/rectifiable arcs along shared boundaries, as in HyperGeography.

Remark 8.5.7 (Nonemptiness and locality)**.** For any $X, Y \in \mathcal{U}_m$, the finite set $T := \mathrm{At}_m(X) \cup \mathrm{At}_m(Y)$ yields

$$Z_0 := \mathrm{Nest}_n(T) \in \mathcal{U}_n, \quad \Upsilon_n(Z_0) = \bigcup_{A \in T} A = \Upsilon_m(X) \cup \Upsilon_m(Y),$$

so $Z_0 \in X \odot_{\mathrm{reg}}^{(m,n)} Y$; hence the operation is well-defined and nonempty. All constructions depend only on atoms intersecting $\Upsilon_m(X) \cup \Upsilon_m(Y)$ (*locality*).

Example 8.5.8 (A $(2,1)$-SuperHyperGeography on the same atom system)**.** Use the same (M, g, d, μ, Σ) and atom set $S = \{A_1, A_2\}$ as in Example 8.5.5. Fix $(m,n) = (2,1)$. A support-finite level-2 object is a nonempty finite set of nonempty finite subfamilies of S. In particular, the canonical nestings

$$X := \mathrm{Nest}_2(\{A_1\}) = \big\{\{A_1\}\big\} \in \mathcal{P}^*_{2,\mathrm{fin}}(S) =: \mathcal{U}_2, \qquad Y := \mathrm{Nest}_2(\{A_2\}) = \big\{\{A_2\}\big\} \in \mathcal{U}_2$$

satisfy

$$\mathrm{At}_2(X) = \{A_1\}, \quad \Upsilon_2(X) = A_1, \qquad \mathrm{At}_2(Y) = \{A_2\}, \quad \Upsilon_2(Y) = A_2$$

by (8.1).

At level $n = 1$, we have $\mathcal{U}_1 = \mathcal{P}^*_{1,\mathrm{fin}}(S) = \mathcal{P}_*(S)$. Define the region superhyperoperation $\odot_{\mathrm{reg}}^{(2,1)} : \mathcal{U}_2 \times \mathcal{U}_2 \to \mathcal{P}(\mathcal{U}_1) \setminus \{\emptyset\}$ by the rule in (8.3):

$$X \odot_{\mathrm{reg}}^{(2,1)} Y = \Big\{ Z \in \mathcal{U}_1 \;\Big|\; \Upsilon_1(Z) = \Upsilon_2(X) \cup \Upsilon_2(Y) \text{ and } \mathrm{At}_1(Z) \subseteq \mathrm{At}_2(X) \cup \mathrm{At}_2(Y) \Big\}.$$

Since $\Upsilon_2(X) \cup \Upsilon_2(Y) = A_1 \cup A_2 = M$ and $\mathrm{At}_2(X) \cup \mathrm{At}_2(Y) = \{A_1, A_2\}$, the only $Z \in \mathcal{U}_1 = \{\{A_1\}, \{A_2\}, \{A_1, A_2\}\}$ satisfying $\Upsilon_1(Z) = M$ is $Z = \{A_1, A_2\}$. Hence

$$X \odot_{\mathrm{reg}}^{(2,1)} Y = \big\{\{A_1, A_2\}\big\} \;\subseteq\; \mathcal{P}(\mathcal{U}_1),$$

which is nonempty and support-finite.

Therefore $\mathbf{SHGeo}^{(2,1)}$ is a concrete (m,n)-SuperHyperGeography, and its region layer realizes a binary $(2,1)$-superhyperoperation returning all admissible level-1 atomic packings whose geometric union matches the level-2 inputs.

8.6 HyperPhilosophy and SuperHyperPhilosophy

HyperPhilosophy models philosophical positions as hyperstates; each concept maps to a set of possible interpretations, enabling multivalued reasoning. SuperHyperPhilosophy iterates that uncertainty hierarchically: interpretations become nested families, supporting meta-level analysis of doctrines, arguments, and paradoxes [460].

Definition 8.6.1 (HyperPhilosophy (hyperstructural consequence system))**.** Let A be a nonempty set of *propositions* and Q a nonempty set of *questions*, with an *addressing map* $\mathrm{adr} : A \to Q$ sending each proposition to the question it addresses. Let M be a nonempty set of *methods* (e.g., formal deduction, probabilistic inference, thought experiments), and R a nonempty set of *contexts* (e.g., background theories, empirical regimes).

A *HyperPhilosophy* is a tuple

$$\mathrm{HP} \;:=\; \big(A,\; Q,\; \mathrm{adr},\; M,\; R,\; \{\, \star_{m,r} \;:\; m \in M,\; r \in R \,\}\big),$$

where for each $(m, r) \in M \times R$ the map

$$\star_{m,r} \;:\; A \times A \;\longrightarrow\; \mathcal{P}(A)$$

is a binary *hyperoperation of argumentative combination* (the set $\star_{m,r}(a, b)$ collects the admissible conclusions obtainable from jointly considering a and b by method m under context r).

For each fixed (m, r) we define the induced *hyperderivation operator*

$$\mathrm{Der}^{\mathrm{H}}_{m,r}(X) \;:=\; X \;\cup\; \bigcup_{a,b \in X} \star_{m,r}(a, b), \qquad X \subseteq A,$$

and the associated *hyperconsequence operator*

$$\mathrm{Cn}^{\mathrm{H}}_{m,r}(\Gamma) \;:=\; \bigcup_{k=0}^{\infty} \Gamma_k, \quad \Gamma_0 := \Gamma, \quad \Gamma_{k+1} := \mathrm{Der}^{\mathrm{H}}_{m,r}(\Gamma_k).$$

For a question $q \in Q$, the set of *acceptable answers under* (m, r) *from premises* Γ is

$$\mathrm{Ans}^{\mathrm{H}}(q \mid \Gamma; m, r) \;:=\; \mathrm{Cn}^{\mathrm{H}}_{m,r}(\Gamma) \;\cap\; \mathrm{adr}^{-1}(\{q\}).$$

Definition 8.6.2 (SuperHyperPhilosophy (graded superhyper-consequence system))**.** Let A be a nonempty set of *propositions*, Q a nonempty set of *questions*, and $\mathrm{adr} : A \to Q$ the *addressing map* sending each proposition to the question it addresses. Let M be a nonempty set of *methods* and R a nonempty set of *contexts*. For each $(m, r) \in M \times R$ fix a family of labelled superhyperoperations

$$\mathcal{O}_{m,r} \;:=\; \Big\{\, \odot_{\ell}^{(m_\ell, n_\ell)} : \big(\mathcal{P}^{\,m_\ell}(A)\big)^{s_\ell} \to \mathcal{P}^{\,n_\ell}(A) \;\Big|\; \ell \in \Lambda_{m,r} \Big\},$$

with signatures $(s_\ell; m_\ell, n_\ell)$ allowed to vary over ℓ.

Write $\mathbf{A}^{\langle k \rangle} := \mathcal{P}^{\,k}(A)$ for the level-k domain. A *graded state* is a sequence $X = (X^{\langle k \rangle})_{k \geq 0}$ with $X^{\langle k \rangle} \subseteq \mathbf{A}^{\langle k \rangle}$. Define the *one-step superhyper-derivation* at (m, r) by

$$\mathrm{Der}^{\mathrm{SH}}_{m,r}(X)^{\langle n \rangle} \;:=\; X^{\langle n \rangle} \;\cup\; \bigcup_{\substack{\ell \in \Lambda_{m,r} \\ n_\ell = n}} \;\bigcup_{(x_1, \ldots, x_{s_\ell}) \in \left(X^{\langle m_\ell \rangle}\right)^{s_\ell}} \odot_{\ell}^{(m_\ell, n_\ell)}(x_1, \ldots, x_{s_\ell}).$$

Starting from initial premises $\Gamma \subseteq A$, set

$$X_0^{\langle 0\rangle} = \Gamma, \qquad X_0^{\langle k\rangle} = \varnothing \ (k \geq 1), \qquad X_{t+1} = \mathrm{Der}_{m,r}^{\mathrm{SH}}(X_t) \quad (t \geq 0).$$

The *graded superhyper-consequence* of Γ is

$$\mathcal{C}_{m,r}^{\mathrm{SH}}(\Gamma)^{\langle n\rangle} \ := \ \bigcup_{t \geq 0} X_t^{\langle n\rangle} \qquad (n \geq 0).$$

To read off base-level conclusions, define the *realization maps* $\mathrm{Real}^{(k)} : \mathbf{A}^{\langle k\rangle} \to \mathcal{P}(A)$ by

$$\mathrm{Real}^{(0)}(a) = \{a\}, \qquad \mathrm{Real}^{(k+1)}(X) = \bigcup_{Y \in X} \mathrm{Real}^{(k)}(Y).$$

The *flattened superhyper-consequence* on A is then

$$\mathcal{C}_{m,r}^{0,\mathrm{SH}}(\Gamma) \ := \ \bigcup_{n \geq 0} \ \bigcup_{x \in \mathcal{C}_{m,r}^{\mathrm{SH}}(\Gamma)^{\langle n\rangle}} \mathrm{Real}^{(n)}(x) \ \subseteq \ A.$$

For each question $q \in Q$, the set of *acceptable answers at* (m, r) *from* Γ is

$$\mathrm{Ans}^{\mathrm{SH}}(q \mid \Gamma; m, r) \ := \ \mathcal{C}_{m,r}^{0,\mathrm{SH}}(\Gamma) \ \cap \ \mathrm{adr}^{-1}(\{q\}).$$

The tuple

$$\mathsf{SHP} \ := \ \big(A, Q, \mathrm{adr}, M, R, \{\mathcal{O}_{m,r}\}_{(m,r) \in M \times R}\big)$$

is called a *SuperHyperPhilosophy*.

Chapter 9

Discussion: HyperStructure, MultiStructure, and MetaStructure

HyperStructure, MultiStructure, and MetaStructure share several common structural features and exhibit partially analogous patterns of organization. In this chapter, we provide a brief yet focused discussion of these frameworks and highlight their points of similarity.

9.1 MultiStructure

A *MultiStructure* on a set H assigns to each m-tuple in H^m a finite multiset of elements drawn from $\mathcal{M}(H)$ [461,462]. Related concepts include MultiGraph [463–465], Multialgebra [466, 467], MultiUncertain Set [468–470], and MultiTopology [471, 472]. MultiPhilosophy extends Philosophy by using multisets: arguments produce bag-valued conclusions, where multiplicities encode strength, redundancy, or evidence across methods and contexts.

Definition 9.1.1 (Finite multisets and basic operations)**.** (cf. [473–476]) Let H be a nonempty set. The collection of *finite multisets* over H is

$$\mathcal{M}(H) \;:=\; \{\, \mu : H \to \mathbb{N} \;\mid\; \#\operatorname{supp}(\mu) < \infty \,\}, \qquad \operatorname{supp}(\mu) := \{\, x \in H : \mu(x) > 0 \,\}.$$

For $\mu, \nu \in \mathcal{M}(H)$ and $k \in \mathbb{N}$ define, pointwise on H,

$$(\mu \oplus \nu)(x) := \mu(x) + \nu(x), \qquad (\mu \sqcup \nu)(x) := \max\{\mu(x), \nu(x)\}, \qquad (k \odot \mu)(x) := k \cdot \mu(x).$$

Write $\mathbf{0} \in \mathcal{M}(H)$ for the zero multiset $\mathbf{0}(x) = 0$. We partially order $\mathcal{M}(H)$ by

$$\mu \preceq \nu \iff \forall x \in H \;\big(\mu(x) \le \nu(x)\big).$$

Then $(\mathcal{M}(H), \preceq)$ is a join-semilattice with join $\sqcup$, and $(\mathcal{M}(H), \oplus, \mathbf{0})$ is a commutative monoid.

Definition 9.1.2 (MultiStructure)**.** [461, 462] Let H be a nonempty set and $\mathcal{I} \subseteq \mathbb{Z}_{>0}$ a set of arities. A *MultiStructure* on H is a pair

$$\mathcal{MS} \;=\; \Big(H,\; \{\; \#^{(m)} : H^m \to \mathcal{M}(H) \;\}_{m \in \mathcal{I}}\Big).$$

For $\mathbf{a} = (a_1, \ldots, a_m) \in H^m$, the multiset $\#^{(m)}(\mathbf{a})$ lists (with multiplicities) the outcomes produced from $\mathbf{a}$.

An *Iterative MultiStructure* builds upon this by iterating the multiset construction through successive levels, yielding operations [461,462,477]. Iterated MultiPhilosophy extends MultiPhilosophy by iterating multiset levels, allowing reasoning over bags-of-bags, capturing hierarchical multiplicity and graded evidence across multiple abstraction layers.

Definition 9.1.3 (Iterated multiset levels)**.** [461,462,477] Let X be a set. Define the levels

$$\mathsf{Multiset}^0(X) := X, \qquad \mathsf{Multiset}^{i+1}(X) := \mathsf{Multiset}\big(\mathsf{Multiset}^i(X)\big) \quad (i \geq 0),$$

so $\mathsf{Multiset}^1(X) = \mathsf{Multiset}(X)$ is the collection of finite multisets on X, $\mathsf{Multiset}^2(X)$ is the collection of finite multisets on $\mathsf{Multiset}(X)$, etc. For $y \in \mathsf{Multiset}^i(X)$ we denote by

$$\boldsymbol{\delta}_y^{\langle i \rangle} \ \in \ \mathsf{Multiset}^{i+1}(X)$$

the *unit multiset at level* $i{+}1$ that assigns multiplicity 1 to y and 0 to all other elements of $\mathsf{Multiset}^i(X)$.

Definition 9.1.4 (Iterative Multi-Structure of order k)**.** [461,462,477] Let H be a nonempty set, fix $k \in \mathbb{Z}_{\geq 1}$, and let $\mathcal{I} \subseteq \mathbb{Z}_{>0}$ be a set of arities. An *Iterative Multi-Structure of order k* consists of a family of *level-wise multi-operations*

$$\mathrm{IMS}^{(k)} \ := \ \Big(H, \ \big\{ \ \#^{(m,i)} : \big(\mathsf{Multiset}^i(H)\big)^m \longrightarrow \mathsf{Multiset}^{i+1}(H) \ \big\}_{m \in \mathcal{I}, \ 0 \leq i < k}\Big).$$

For inputs $x_1, \ldots, x_m \in \mathsf{Multiset}^i(H)$, the value $\#^{(m,i)}(x_1, \ldots, x_m)$ is an *output object* at the next level $\mathsf{Multiset}^{i+1}(H)$.

Example 9.1.5 (Security incident triage as a MultiStructure and an Iterative MultiStructure)**.** Consider a cybersecurity operations center (SOC) that performs *incident triage* by correlating alerts from multiple detectors (SIEM/IDS/EDR) and mapping them to candidate *attack hypotheses* and *mitigation actions*.

Step 1 (MultiStructure level). Let H be a finite set of SOC tokens that includes alert types, hypotheses, and actions, e.g.,

$$H = \{\mathsf{FL}, \mathsf{GA}, \mathsf{OB}, \mathsf{BF}, \mathsf{CS}, \mathsf{C2}, \mathsf{BL}, \mathsf{PR}\},$$

where: FL = "many failed logins", GA = "geo-anomalous login", OB = "suspicious outbound traffic", BF = "brute-force", CS = "credential-stuffing", $\mathsf{C2}$ = "malware C2", BL = "block IP", PR = "force password reset". Take $\mathcal{I} = \{2\}$ (binary correlation rules). Define a MultiStructure operation $\#^{(2)} : H^2 \to \mathcal{M}(H)$ by letting multiplicities count how many independent correlation rules (or analytic modules) support the same output token. Using the unit multiset $\boldsymbol{\delta}_x^{\langle 0 \rangle} \in \mathsf{Multiset}^1(H) = \mathcal{M}(H)$, set for instance:

$$\#^{(2)}(\mathsf{FL}, \mathsf{GA}) := 2 \odot \boldsymbol{\delta}_{\mathsf{CS}}^{\langle 0 \rangle} \ \oplus \ \boldsymbol{\delta}_{\mathsf{BF}}^{\langle 0 \rangle} \ \oplus \ \boldsymbol{\delta}_{\mathsf{PR}}^{\langle 0 \rangle}.$$

This encodes that two independent detectors/rules indicate CS, one indicates BF, and one suggests the action PR. Likewise,

$$\#^{(2)}(\mathsf{OB}, \mathsf{GA}) := \boldsymbol{\delta}_{\mathsf{C2}}^{\langle 0 \rangle} \ \oplus \ \boldsymbol{\delta}_{\mathsf{BL}}^{\langle 0 \rangle},$$

meaning that the same correlation of signals supports a malware C2 hypothesis and recommends blocking.

Step 2 (Iterated MultiStructure level). Now suppose the SOC runs three subsystems that each output a *bag of hypotheses/actions* (i.e., an element of $\mathsf{Multiset}^1(H)$), and the SOC manager wants a *bag-of-bags* (i.e., an element of $\mathsf{Multiset}^2(H) = \mathsf{Multiset}(\mathsf{Multiset}(H))$) that records how many subsystems produced the same incident narrative.

Let

$$B_{\mathsf{SIEM}} := \#^{(2,0)}(\mathsf{FL}, \mathsf{GA}), \qquad B_{\mathsf{IDS}} := \#^{(2,0)}(\mathsf{FL}, \mathsf{GA}), \qquad B_{\mathsf{EDR}} := \#^{(2,0)}(\mathsf{OB}, \mathsf{GA}),$$

so $B_{\mathsf{SIEM}} = B_{\mathsf{IDS}}$ (two subsystems agree) while B_{EDR} differs. Define an order-$k = 2$ iterative MultiStructure by including, for $i = 1$, an aggregation operation

$$\#^{(3,1)} : \big(\mathsf{Multiset}^1(H)\big)^3 \longrightarrow \mathsf{Multiset}^2(H),$$

and set (using the level-2 unit multiset $\boldsymbol{\delta}^{\langle 1\rangle}_B \in \mathsf{Multiset}^2(H)$ for $B \in \mathsf{Multiset}^1(H)$):

$$\#^{(3,1)}(B_{\mathsf{SIEM}}, B_{\mathsf{IDS}}, B_{\mathsf{EDR}}) := 2 \odot \boldsymbol{\delta}^{\langle 1\rangle}_{B_{\mathsf{SIEM}}} \ \oplus \ \boldsymbol{\delta}^{\langle 1\rangle}_{B_{\mathsf{EDR}}} \ \in \ \mathsf{Multiset}^2(H).$$

Hence the output is a bag-of-bags in which the narrative B_{SIEM} appears with multiplicity 2 (two subsystems support it), while B_{EDR} appears with multiplicity 1. This is a concrete instance where a MultiStructure captures *within-level* evidence multiplicity, and an Iterative MultiStructure captures *hierarchical* multiplicity across multiple abstraction layers (bags of hypotheses, and bags of such bags).

9.2 MetaStructure (Recall)

A MetaStructure is a higher-order framework where operations act on entire structures, producing new structures while preserving isomorphisms(cf. [376, 478]).

Definition 9.2.1 (Signature and Σ-structures)**.** (cf. [376, 478, 479]) A (single-sorted, finitary) *signature* is a tuple

$$\Sigma \ = \ \big(\mathsf{Func}, \mathsf{Rel}, \mathrm{ar}_{\mathsf{Func}}, \mathrm{ar}_{\mathsf{Rel}}\big),$$

where Func and Rel are sets of function and relation symbols, and ar gives their arities. A Σ*–structure* is a tuple

$$\mathbf{C} \ = \ \big(H, \ (f^{\mathbf{C}})_{f\in\mathsf{Func}}, \ (R^{\mathbf{C}})_{R\in\mathsf{Rel}}\big),$$

with nonempty carrier H, each $f^{\mathbf{C}} : H^m \to H$ for $m = \mathrm{ar}(f)$, and each $R^{\mathbf{C}} \subseteq H^r$ for $r = \mathrm{ar}(R)$. Let Str_Σ be the class of all Σ–structures.

Definition 9.2.2 (MetaStructure over a fixed signature)**.** (cf. [478]) Given Σ as in Theorem 9.2.1, a *MetaStructure* is a pair

$$\mathbb{M} \ = \ \big(U, \ (\Phi_\ell)_{\ell\in\Lambda}\big),$$

where:

- $U \subseteq \mathrm{Str}_\Sigma$ is a nonempty collection of Σ–structures;
- each *meta–operation* $\Phi_\ell : U^{k_\ell} \to U$ (meta-arity $k_\ell \in \mathbb{N}$) is specified by uniform constructors on carriers and symbol-interpretations.

Isomorphism invariance (naturality): if $\alpha_i : \mathbf{C}_i \cong \mathbf{D}_i$ in Str_Σ, then $\Phi_\ell(\alpha_1, \dots, \alpha_{k_\ell}) : \Phi_\ell(\mathbf{C}_1, \dots, \mathbf{C}_{k_\ell}) \cong \Phi_\ell(\mathbf{D}_1, \dots, \mathbf{D}_{k_\ell})$.

An *Iterated MetaStructure* is obtained by repeatedly applying the MetaStructure construction, thereby forming successive levels where "structures of structures" build a hierarchical tower (cf. [376, 478]).

Definition 9.2.3 (Iterated MetaStructure of depth t)**.** (cf. [478]) For $t \in \mathbb{N}$, an *Iterated MetaStructure of depth t* over Σ is a MetaStructure $\mathfrak{M}^{(t)}$ obtained by t iterations of a lifting procedure. When $s < t$, we *lift* a height–s MetaStructure $\mathfrak{M}^{(s)} = (U^{(s)}, \{\odot_i\}, \{\mathcal{S}_j\})$ to height t by

$$\iota_{s\to t} : \; U^{(s)} \xrightarrow{\mathsf{U}_\Sigma^{t-s}} U^{(t)} \;:=\; \mathsf{U}_\Sigma^{t-s}\big(U^{(s)}\big),$$

and, for each meta–operation $\odot_i : (\mathsf{E}_\Sigma^{m_i})^{k_i} \to \mathcal{P}^{\,n_i}(\mathsf{E}_\Sigma^{n_i})$, define its lift by

$$\odot_i^{\uparrow} : \big(\mathsf{E}_\Sigma^{m_i+t-s}\big)^{k_i} \longrightarrow \mathcal{P}^{\,n_i}\big(\mathsf{E}_\Sigma^{n_i+t-s}\big), \qquad \odot_i^{\uparrow}\big(\mathsf{U}_\Sigma^{t-s}(x_1), \dots, \mathsf{U}_\Sigma^{t-s}(x_{k_i})\big) := \mathsf{U}_\Sigma^{t-s}\big(\odot_i(x_1, \dots, x_{k_i})\big),$$

and analogously for relations $\mathcal{S}_j^{\uparrow} := \big(\mathsf{U}_\Sigma^{t-s}\big)^{\times \ell_j}(\mathcal{S}_j)$.

Based on the above, we present a concise comparison of MetaStructure, MultiStructure, and HyperStructure in Table 9.1. We also present a concise comparison of Iterated MetaStructure, Iterated MultiStructure, and SuperHyperStructure in Table 9.2.

Table 9.1: Concise comparison of MetaStructure, MultiStructure, and HyperStructure.

Concept	**Carrier / Objects**	**Primitive operation type**	**Output semantics**
MetaStructure	$U \subseteq \mathrm{Str}_\Sigma$ (structures)	meta-operations $\Phi : U^k \to U$	structure-level construction; isomorphism-invariant
MultiStructure	H (atoms)	multi-operations $\#^{(m)} : H^m \to \mathcal{M}(H)$	finite multiset (bag) of outcomes with multiplicities
HyperStructure	S (atoms) or $\mathcal{P}(S)$ (sets)	hyperoperations $\circ : S^m \to \mathcal{P}(S)$	set-valued outcomes (nonempty subset, typically)

Table 9.2: Concise comparison of iterated constructions and SuperHyperStructure.

Concept	**Levels / Carrier at level t**	**Lifted operation type**	**Key idea**
Iterated MetaStructure	$U^{(t)}$ = structures-of-$\cdots$-of-structures	$\Phi^{(t)} : (U^{(t)})^k \to U^{(t)}$	compose uniform constructors across meta-levels
Iterated MultiStructure	$\mathsf{Multiset}^t(H)$ (bags-of-$\cdots$-bags)	$\#^{(m,t)} : (\mathsf{Multiset}^t(H))^m \to \mathsf{Multiset}^{t+1}(H)$	iterate multiplicity/aggregation hierarchically
SuperHyperStructure	$\mathcal{P}^t(S)$ (nested sets)	$\star : (\mathcal{P}^m(S))^k \to \mathcal{P}^n(S)$	iterate set-valued structure via powerset hierarchy

Example 9.2.4 (Health-information exchange as a MetaStructure and an iterated MetaStructure)**.** We describe a concrete real-world setting from *health-information exchange (HIE)* and show how it fits both Theorem 9.2.2 and Theorem 9.2.3.

(1) A fixed single-sorted signature Σ for EHR graphs. Let the carrier H represent all "records" (patients, visits, lab-events, etc.) together with a distinguished null record $\perp$. Take a finitary single-sorted signature

$$\Sigma = \big(\mathsf{Func}, \mathsf{Rel}, \mathrm{ar}_{\mathsf{Func}}, \mathrm{ar}_{\mathsf{Rel}}\big)$$

with (for simplicity) the following symbols:

$$\mathsf{Func} = \{\ \mathsf{patOf}, \mathsf{visitOf}, \mathsf{labOf}\ \}, \qquad \mathsf{Rel} = \{\ \mathsf{Patient}, \mathsf{Visit}, \mathsf{Lab}\ \},$$

where $\mathrm{ar}(\mathsf{patOf}) = \mathrm{ar}(\mathsf{visitOf}) = \mathrm{ar}(\mathsf{labOf}) = 1$ and each relation is unary. Intuitively, $\mathsf{Patient}(x)$ means "x is a patient-record", etc. The functions encode typed pointers, totalized using $\perp$: $\mathsf{patOf}(v)$ returns the patient-record attached to a visit-record v (and $\perp$ if v is not a visit); $\mathsf{visitOf}(\ell)$ returns the visit-record attached to a lab-record ℓ (and $\perp$ otherwise); $\mathsf{labOf}(v)$ returns a canonical lab-record associated to v if one exists (else $\perp$). (Any practical EHR has many more symbols; this toy signature is enough to illustrate the meta-level construction.)

A concrete hospital database snapshot is then a Σ-structure

$$\mathbf{C} = \big(H,\ (\mathsf{patOf}^{\mathbf{C}}, \mathsf{visitOf}^{\mathbf{C}}, \mathsf{labOf}^{\mathbf{C}}),\ (\mathsf{Patient}^{\mathbf{C}}, \mathsf{Visit}^{\mathbf{C}}, \mathsf{Lab}^{\mathbf{C}})\big) \in \mathrm{Str}_{\Sigma},$$

where the unary relations partition the intended record-types (up to the presence of $\perp$) and the functions respect types (e.g., if $\mathsf{Visit}^{\mathbf{C}}(v)$ then either $\mathsf{Patient}^{\mathbf{C}}(\mathsf{patOf}^{\mathbf{C}}(v))$ or $\mathsf{patOf}^{\mathbf{C}}(v) = \perp$).

(2) MetaStructure: a regional SOC/HIE integration layer. Let $U \subseteq \mathrm{Str}_{\Sigma}$ be the collection of all nightly EHR snapshots from hospitals in one region: each $\mathbf{C} \in U$ is one hospital's database state (same signature, different carrier and interpretations).

Define two meta-operations that are standard in HIE practice:

(a) **Federated merge (deduplicating union).** A meta-operation $\Phi_{\mathsf{merge}} : U^2 \to U$ takes two hospital snapshots $\mathbf{C}_1, \mathbf{C}_2 \in U$ and returns the *regional merged view*

$$\Phi_{\mathsf{merge}}(\mathbf{C}_1, \mathbf{C}_2) =: \mathbf{C}_{12} \in U$$

constructed uniformly as follows.

Carrier constructor: start from the disjoint union $H_{12} := H_1 \bigsqcup H_2$ (with $\perp$ identified), and then quotient by an equivalence relation $\sim$ that identifies records deemed to correspond to the same real-world entity (e.g., same hashed national patient identifier, same visit key under an agreed reconciliation rule). Write $[x]$ for the $\sim$-class of x and set $H_{12} := (H_1 \bigsqcup H_2)/\!\sim$.

Interpretation constructor: define, for each unary relation symbol $R \in \{\mathsf{Patient}, \mathsf{Visit}, \mathsf{Lab}\}$,

$$R^{\mathbf{C}_{12}} := \{\, [x] \in H_{12}\ :\ x \in R^{\mathbf{C}_1} \cup R^{\mathbf{C}_2}\,\},$$

and for each function symbol $f \in \{\mathsf{patOf}, \mathsf{visitOf}, \mathsf{labOf}\}$,

$$f^{\mathbf{C}_{12}}([x]) := [\, f^{\mathbf{C}_i}(x)\,] \quad \text{whenever } x \in H_i\ (i = 1, 2),$$

which is well-defined because the reconciliation $\sim$ is chosen to be compatible with the pointers (the standard "entity resolution" consistency condition in data integration).

(b) **Anonymization/pseudonymization.** A meta-operation $\Phi_{\mathsf{anon}} : U \to U$ replaces all patient-identifying records by pseudonyms, implemented at the structure level by a uniform carrier map $\pi : H \to H'$ that collapses/renames identifiers while preserving the typed-pointer pattern:

$$\Phi_{\mathsf{anon}}(\mathbf{C}) =: \mathbf{C}' \in U, \qquad H' = \pi(H), \quad R^{\mathbf{C}'} = \pi(R^{\mathbf{C}}), \quad f^{\mathbf{C}'}(\pi(x)) = \pi(f^{\mathbf{C}}(x)).$$

Hence $\mathbb{M}^{(0)} := (U, \Phi_{\mathsf{merge}}, \Phi_{\mathsf{anon}})$ is a MetaStructure in the sense of Theorem 9.2.2.

Isomorphism invariance (why it holds in practice). If $\alpha : \mathbf{C} \cong \mathbf{D}$ is an isomorphism in Str_Σ (a renaming of record IDs that preserves $\mathsf{Patient}, \mathsf{Visit}, \mathsf{Lab}$ and commutes with $\mathsf{patOf}, \mathsf{visitOf}, \mathsf{labOf}$), then the constructors above do not depend on concrete IDs, only on the incidence pattern and reconciliation criteria. Therefore,

$$\mathbf{C}_1 \cong \mathbf{D}_1,\ \mathbf{C}_2 \cong \mathbf{D}_2 \implies \Phi_{\mathsf{merge}}(\mathbf{C}_1, \mathbf{C}_2) \cong \Phi_{\mathsf{merge}}(\mathbf{D}_1, \mathbf{D}_2), \qquad \Phi_{\mathsf{anon}}(\mathbf{C}) \cong \Phi_{\mathsf{anon}}(\mathbf{D}),$$

which is exactly the naturality requirement in Theorem 9.2.2.

(3) Iterated MetaStructure: regional exchanges aggregated into a national exchange. Now consider multiple regional HIE operators (Region A, Region B, …), each maintaining its own MetaStructure $\mathbb{M}_{\mathrm{A}}^{(0)}, \mathbb{M}_{\mathrm{B}}^{(0)}, \ldots$ over the same Σ. At the next level, a national HIE forms *structures of structures* by treating each *regional integrated snapshot* as a first-class object and then integrating those objects.

Concretely, let $U^{(1)} \subseteq \mathrm{Str}_\Sigma$ be the collection of *regional integrated snapshots* obtained by applying Φ_{merge} repeatedly inside each region (and possibly Φ_{anon}), and define meta-operations on $U^{(1)}$ such as:

$$\Phi^{(1)}_{\mathsf{nat_merge}} : (U^{(1)})^2 \to U^{(1)} \quad \text{(merge two } \textit{regions}\text{)},$$

$$\Phi^{(1)}_{\mathsf{policy}} : U^{(1)} \to U^{(1)} \quad \text{(apply a uniform national policy)}.$$

These are specified by the *same kind of constructors* as before (disjoint union / reconciliation for merge, uniform quotient/renaming for policy), but now the inputs are already *structures produced by meta-operations*. This yields a height–1 MetaStructure

$$\mathbb{M}^{(1)} := \big(U^{(1)}, \Phi^{(1)}_{\mathsf{nat_merge}}, \Phi^{(1)}_{\mathsf{policy}}\big).$$

Iterating once more gives a depth-$t = 2$ tower (e.g., international federation of national exchanges), which matches Theorem 9.2.3 conceptually: lower-level objects (hospital snapshots) are lifted to higher levels (regional snapshots, then national snapshots), and the merge/policy operations lift accordingly to act on these higher-level objects while remaining isomorphism-invariant.

Hospital EHR instances are Σ-structures; a regional HIE defines a MetaStructure by meta-operations "merge" and "anonymize" on a nonempty class $U \subseteq \mathrm{Str}_\Sigma$; and a national HIE forms an Iterated MetaStructure by repeating the same style of structure-constructors on the outputs of the previous level.

Chapter 10

New Structure: Sheaf HyperStructure and SuperHyperStructure

In this chapter, we introduce *Sheaf HyperStructures* and *Sheaf SuperHyperStructures* as new notions of HyperStructures and SuperHyperStructures, and we investigate their fundamental properties.

10.1 Sheaf HyperStructure

A Sheaf HyperStructure is a sheaf that assigns a local hyperoperation to each open set, compatible with restrictions, so local multi-valued compositions glue consistently.

Definition 10.1.1 (Sheaf HyperStructure)**.** Let (X, τ) be a topological space. Let $\mathcal{F}$ be a sheaf of sets on X, i.e., for every open set $U \in \tau$ a set $\mathcal{F}(U)$ is given, and for every inclusion of open sets $V \subseteq U$ a restriction map

$$\rho_{U,V} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$$

is given, such that the presheaf identities hold $\rho_{U,U} = \mathrm{id}_{\mathcal{F}(U)}$ and $\rho_{U,W} = \rho_{V,W} \circ \rho_{U,V}$ for $W \subseteq V \subseteq U$, and moreover the sheaf axioms (locality and gluing) hold.

For each open set $U \in \tau$, assume we are given a (set-valued) binary operation

$$\circ_U : \mathcal{F}(U) \times \mathcal{F}(U) \longrightarrow \mathcal{P}\big(\mathcal{F}(U)\big),$$

called the *local hyperoperation on* U. For any map $f : A \to B$, write $f^\sharp : \mathcal{P}(A) \to \mathcal{P}(B)$ for the induced map $f^\sharp(S) := \{f(s) \mid s \in S\}$.

We say that $(\mathcal{F}, \{\circ_U\}_{U \in \tau})$ is a *sheaf hyperstructure on* X if, for every inclusion $V \subseteq U$ of open sets and all $a, b \in \mathcal{F}(U)$, the following *restriction-compatibility* holds:

$$\rho_{U,V}^\sharp\big(a \circ_U b\big) \;\subseteq\; \rho_{U,V}(a) \circ_V \rho_{U,V}(b). \tag{10.1}$$

If equality holds in (10.1) for all $V \subseteq U$ and all a, b, we call it a *strict sheaf hyperstructure*.

Table 10.1: Concise comparison of (global) HyperStructures and Sheaf HyperStructures.

Aspect	**HyperStructure**	**Sheaf HyperStructure (Def. 10.1.1)**
Underlying setting	A single carrier set H.	A topological space (X, τ) and a sheaf of sets $\mathcal{F}$; carriers vary with $U \in \tau$ as $\mathcal{F}(U)$.
Operation(s)	One (or finitely many) global hyperoperation(s), e.g. $\circ : H \times H \to \mathcal{P}(H)$.	A family of *local* hyperoperations $\circ_U : \mathcal{F}(U) \times \mathcal{F}(U) \to \mathcal{P}(\mathcal{F}(U))$ for each open U.
Local-to-global mechanism	None built in (global data are primitive).	Built in via restriction maps $\rho_{U,V}$ and sheaf axioms (locality + gluing) for sections.
Compatibility / coherence	Coherence expressed only by algebraic axioms on H (associativity-like laws, etc., depending on the type).	Restriction-compatibility: $\rho^{\sharp}_{U,V}(a \circ_U b) \subseteq \rho_{U,V}(a) \circ_V \rho_{U,V}(b)$ for $V \subseteq U$; equality gives the strict case.
Meaning of "element"	Element $x \in H$ is a global object.	Section $s \in \mathcal{F}(U)$ is a *context-dependent* (local) object over U.
Notion of morphism (typical)	A map $f : H \to K$ preserving hyperoperations (often $f^{\sharp}(x \circ y) \subseteq f(x) \diamond f(y)$).	A morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ commuting with restrictions and (typically) locally preserving hyperoperations on each U.
Information interpretation	Multi-valued composition at a single global scale.	Multi-valued composition that can vary by region/context and must be consistent under restriction and gluing.

For reference, a comparison of (global) HyperStructures and Sheaf HyperStructures is presented in Table 10.1.

Example 10.1.2 (Real-world example: incident response across microservices)**.** Consider a large-scale cloud service composed of many microservices deployed across regions. Let X be the (finite) set of services and regions, and equip X with a topology whose open sets $U \subseteq X$ represent *operational scopes* (e.g., "all components in Region 1", "the payments pipeline", or "the set of services touched by a given customer journey"). Thus a larger open set corresponds to a broader scope of observability and coordination.

Sheaf of local states. For each open set $U \in \tau$, let $\mathcal{F}(U)$ be the set of feasible *local incident-response states* on U. Concretely, an element $s \in \mathcal{F}(U)$ may encode the current health and mitigation configuration on U, such as

$$s = \big(\text{alert level},\ \text{traffic-shift policy},\ \text{feature-flag vector},\ \text{rate-limit plan}\big)\big|_U.$$

If $V \subseteq U$, the restriction map $\rho_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$ forgets information outside V, i.e., it restricts the configuration on U to the smaller scope V. The sheaf gluing axiom expresses a standard operational principle: if we choose configurations $s_i \in \mathcal{F}(U_i)$ on scopes $U = \bigcup_i U_i$ that agree on overlaps $U_i \cap U_j$, then there exists a unique global configuration $s \in \mathcal{F}(U)$ consistent with all s_i.

Local hyperoperation (multi-valued composition). In incident response, combining two local actions is often *non-deterministic* because multiple mitigation plans can be valid depending on latency, load, and downstream dependencies. Define, for each $U \in \tau$, a hyperoperation

$$\circ_U : \mathcal{F}(U) \times \mathcal{F}(U) \to \mathcal{P}\big(\mathcal{F}(U)\big)$$

by letting $a \circ_U b$ be the set of all feasible states obtained by executing (or attempting to execute) the mitigation encoded by a together with the mitigation encoded by b on scope U. For instance, if a means "shift 30% traffic to Region 2" and b means "enable fallback mode", then $a \circ_U b$ may contain several outcomes:

- fallback succeeds and traffic-shift is stable,
- fallback succeeds but traffic-shift is partially rolled back due to elevated error rates,
- fallback is enabled only for a subset of endpoints due to dependency constraints.

Restriction-compatibility (local-to-local consistency). If $V \subseteq U$, then restricting a combined mitigation outcome on U to V must be feasible as a mitigation outcome computed directly on the smaller scope V:

$$\rho^{\sharp}_{U,V}\big(a \circ_U b\big) \;\subseteq\; \rho_{U,V}(a) \circ_V \rho_{U,V}(b).$$

Operationally, this says: any global feasible outcome, when projected to a subsystem V, is still an outcome that the subsystem could realize under the corresponding restricted actions. Hence $(\mathcal{F}, \{\circ_U\}_{U\in\tau})$ forms a *Sheaf HyperStructure* capturing multi-valued incident-response composition that glues consistently from local scopes to global scopes.

Theorem 10.1.3 (Well-definedness of Sheaf HyperStructures)**.** *Let (X, τ) be a topological space and let $\mathcal{F}$ be a sheaf of sets on X with restriction maps $\rho_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$ for $V \subseteq U$. Assume that for each open set $U \in \tau$ we are given a set-valued binary operation*

$$\circ_U : \mathcal{F}(U) \times \mathcal{F}(U) \longrightarrow \mathcal{P}\big(\mathcal{F}(U)\big).$$

For any map $f : A \to B$, define $f^{\sharp} : \mathcal{P}(A) \to \mathcal{P}(B)$ by $f^{\sharp}(S) := \{f(s) \mid s \in S\}$.

Suppose the restriction-compatibility *condition holds: for every inclusion $V \subseteq U$ and all $a, b \in \mathcal{F}(U)$,*

$$\rho^{\sharp}_{U,V}\big(a \circ_U b\big) \;\subseteq\; \rho_{U,V}(a) \circ_V \rho_{U,V}(b). \tag{10.2}$$

Then the data $(\mathcal{F}, \{\circ_U\}_{U\in\tau})$ is well-defined in the following sense:

1. *For every $U \in \tau$, the pair $\big(\mathcal{F}(U), \circ_U\big)$ is a hyperstructure (a hypergroupoid), i.e., $\circ_U$ is a hyperoperation on the carrier set $\mathcal{F}(U)$.*
2. *For every inclusion $V \subseteq U$, the restriction map $\rho_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$ is a* (weak) homomorphism of hyperstructures*, meaning it satisfies* (10.2)*.*

3. The compatibility condition is coherent under iterated restriction*: for any $W \subseteq V \subseteq U$ and any $a, b \in \mathcal{F}(U)$, one has*

$$\rho^{\sharp}_{U,W}\big(a \circ_U b\big) \ \subseteq \ \rho_{U,W}(a) \circ_W \rho_{U,W}(b),$$

so the notion does not depend on whether one restricts in one step or in several steps.

Proof. **(1)** Fix $U \in \tau$. By assumption, $\circ_U : \mathcal{F}(U) \times \mathcal{F}(U) \to \mathcal{P}(\mathcal{F}(U))$ is a well-typed set-valued binary operation on the carrier set $\mathcal{F}(U)$. This is exactly the defining requirement for a hyperoperation on $\mathcal{F}(U)$; hence $\big(\mathcal{F}(U), \circ_U\big)$ is a hyperstructure (hypergroupoid).

(2) Let $V \subseteq U$. The restriction map $\rho_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$ is already part of the sheaf data, hence is a well-defined function between the carrier sets. The induced map $\rho^{\sharp}_{U,V} : \mathcal{P}(\mathcal{F}(U)) \to \mathcal{P}(\mathcal{F}(V))$ is well-defined by construction of $f^{\sharp}$ for any function f. The inclusion (10.2) is precisely the homomorphism condition, so $\rho_{U,V}$ is a (weak) homomorphism of hyperstructures.

(3) We first note a basic functoriality property of the "$\sharp$" construction: for any maps $A \xrightarrow{f} B \xrightarrow{g} C$ and any $S \subseteq A$,

$$(g \circ f)^{\sharp}(S) = \{(g \circ f)(s) \mid s \in S\} = \{g(f(s)) \mid s \in S\} = g^{\sharp}\big(f^{\sharp}(S)\big).$$

Hence $(g \circ f)^{\sharp} = g^{\sharp} \circ f^{\sharp}$, and also $(\mathrm{id}_A)^{\sharp} = \mathrm{id}_{\mathcal{P}(A)}$.

Now take open sets $W \subseteq V \subseteq U$ and elements $a, b \in \mathcal{F}(U)$. Since $\mathcal{F}$ is a sheaf (in particular a presheaf), its restriction maps satisfy

$$\rho_{U,W} = \rho_{V,W} \circ \rho_{U,V}.$$

Applying the functoriality of "$\sharp$" gives

$$\rho^{\sharp}_{U,W} = (\rho_{V,W} \circ \rho_{U,V})^{\sharp} = \rho^{\sharp}_{V,W} \circ \rho^{\sharp}_{U,V}.$$

Using (10.2) for $V \subseteq U$ yields

$$\rho^{\sharp}_{U,V}\big(a \circ_U b\big) \subseteq \rho_{U,V}(a) \circ_V \rho_{U,V}(b).$$

Applying $\rho^{\sharp}_{V,W}$ to both sides (monotonicity of direct image on subsets) gives

$$\rho^{\sharp}_{V,W}\Big(\rho^{\sharp}_{U,V}(a \circ_U b)\Big) \subseteq \rho^{\sharp}_{V,W}\Big(\rho_{U,V}(a) \circ_V \rho_{U,V}(b)\Big).$$

The left-hand side is $\rho^{\sharp}_{U,W}(a \circ_U b)$ by the identity above. On the right-hand side, applying (10.2) for $W \subseteq V$ with $a' = \rho_{U,V}(a)$ and $b' = \rho_{U,V}(b)$ yields

$$\rho^{\sharp}_{V,W}\Big(\rho_{U,V}(a) \circ_V \rho_{U,V}(b)\Big) \subseteq \rho_{V,W}\big(\rho_{U,V}(a)\big) \circ_W \rho_{V,W}\big(\rho_{U,V}(b)\big).$$

Finally, $\rho_{V,W} \circ \rho_{U,V} = \rho_{U,W}$, so the right-hand side equals $\rho_{U,W}(a) \circ_W \rho_{U,W}(b)$. Combining the inclusions proves

$$\rho^{\sharp}_{U,W}\big(a \circ_U b\big) \subseteq \rho_{U,W}(a) \circ_W \rho_{U,W}(b),$$

which is the desired coherence under iterated restriction. Therefore the notion of Sheaf HyperStructure is well-defined and internally consistent. □

10.2 Sheaf SuperHyperStructure

A Sheaf SuperHyperStructure is a sheaf whose local operations take values in iterated powersets, remain restriction-compatible, and encode hierarchical multi-valued composition globally.

Definition 10.2.1 (Sheaf SuperHyperStructure)**.** Let (X, τ) be a topological space and let $\mathcal{F}$ be a sheaf of sets on X with restriction maps $\rho_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$ for $V \subseteq U$. Fix an integer $n \in \mathbb{N}$, and define the iterated powerset functor $\mathcal{P}^n$ by

$$\mathcal{P}^0(S) := S, \qquad \mathcal{P}^{k+1}(S) := \mathcal{P}\big(\mathcal{P}^k(S)\big) \quad (k \geq 0).$$

For any map $f : A \to B$, define inductively the induced maps

$$f^{(0)} := f, \qquad f^{(k+1)} := \big(f^{(k)}\big)^\sharp : \mathcal{P}^{k+1}(A) \to \mathcal{P}^{k+1}(B).$$

In particular, for $V \subseteq U$ we obtain $(\rho_{U,V})^{(n)} : \mathcal{P}^n(\mathcal{F}(U)) \to \mathcal{P}^n(\mathcal{F}(V))$.

For each open set $U \in \tau$, assume we are given an n-level superhyperoperation

$$\star_U : \mathcal{F}(U) \times \mathcal{F}(U) \longrightarrow \mathcal{P}^n\big(\mathcal{F}(U)\big).$$

We say that $(\mathcal{F}, \{\star_U\}_{U\in\tau})$ is an *n-sheaf superhyperstructure on X* if, for every inclusion $V \subseteq U$ and all $a, b \in \mathcal{F}(U)$, the following *n-level restriction-compatibility* holds:

$$(\rho_{U,V})^{(n)}\big(a \star_U b\big) \;\subseteq\; \rho_{U,V}(a) \star_V \rho_{U,V}(b). \tag{10.3}$$

If equality holds in (10.3) for all $V \subseteq U$ and all a, b, we call it a *strict n-sheaf superhyperstructure*.

For reference, a comparison of (global) SuperHyperStructures and n-sheaf SuperHyperStructures is presented in Table 10.2.

Example 10.2.2 (Real-world example: hierarchical clinical decision support across hospital units)**.** Consider a hospital network consisting of multiple wards (ICU, ER, general medicine) distributed across campuses. Let X be the set of clinical units and care contexts, and endow X with a topology whose open sets $U \in \tau$ represent *coordination scopes* (e.g., "all beds in the ICU", "the cardiology service line", or "all units on Campus A"). Larger open sets correspond to broader organizational scopes.

Sheaf of local clinical states. For each open set $U \in \tau$, let $\mathcal{F}(U)$ be the set of feasible *local clinical states* on U. An element $s \in \mathcal{F}(U)$ can be viewed as a bundle of structured patient- and unit-level information available at scope U, for example

$$s = \big(\text{vital-sign trend summaries},\ \text{lab panels},\ \text{contraindication flags},\ \text{resource availability}\big)\big|_U.$$

If $V \subseteq U$, the restriction map $\rho_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$ restricts the data and plans to the smaller scope, discarding information external to V. The sheaf gluing axiom expresses that if clinicians specify mutually consistent local states on a cover $U = \bigcup_i U_i$ (agreeing on overlaps), then there is a unique global state on U that realizes these locals.

Why an iterated powerset output is natural. Clinical decision support is inherently hierarchical:

Table 10.2: Concise comparison of (global) SuperHyperStructures and n-sheaf SuperHyperStructures.

Aspect	**SuperHyperStructure (global)**	**n-Sheaf SuperHyperStructure (Def. 10.2.1)**
Underlying setting	A single carrier set S (often built from an iterated powerset universe).	A topological space (X, τ) and a sheaf of sets $\mathcal{F}$; carriers vary with $U \in \tau$ as $\mathcal{F}(U)$.
Output level / nesting	A fixed level n (or a profile such as (m, n)): operations output nested sets, e.g. $\star : S \times S \to \mathcal{P}^n(S)$.	A fixed n and *local* operations $\star_U : \mathcal{F}(U) \times \mathcal{F}(U) \to \mathcal{P}^n(\mathcal{F}(U))$.
Local-to-global mechanism	None built in (global data are primitive).	Built in via restriction maps $\rho_{U,V}$ and sheaf axioms (locality + gluing) for sections.
Compatibility / coherence	Coherence expressed only by superhyper-algebraic axioms on S (associativity-like, distributive-like, etc., depending on the type).	n-level restriction-compatibility: $(\rho_{U,V})^{(n)}(a \star_U b) \subseteq \rho_{U,V}(a) \star_V \rho_{U,V}(b)$; equality gives the strict case.
Meaning of "element"	Element $x \in S$ is a global object (or a global nested object).	Section $s \in \mathcal{F}(U)$ is a context-dependent local object; nesting appears in the *output* $\mathcal{P}^n(\mathcal{F}(U))$.
Induced maps / functoriality	Typically uses $f^{(n)}$ (iterated $\sharp$) for morphisms $f : S \to T$ to compare $\mathcal{P}^n$-outputs.	Canonically uses $(\rho_{U,V})^{(n)}$ induced by restrictions to transport n-level outputs between opens.
Notion of morphism (typical)	A map $f : S \to T$ preserving n-level operation, often $f^{(n)}(x \star y) \subseteq f(x) \diamond f(y)$.	A sheaf morphism $\varphi : \mathcal{F} \to \mathcal{G}$ commuting with restrictions and locally preserving $\star_U$ on each U.
Information interpretation	Multi-valued *nested* composition at a single global scale.	Nested composition that can vary by region/context and must be consistent under restriction and gluing across X.

- At the first level, the system proposes a *set of candidate interventions* (e.g., oxygen titration, fluid bolus, antibiotics, imaging).

- Each candidate intervention expands into a *set of protocol variants* (dose ranges, timing windows, monitoring requirements).

- Each variant expands into a *set of contingency branches* (what-if actions if blood pressure drops, if renal function changes, if an allergy is discovered).

This yields a nested "set-of-sets-of-sets" structure, naturally modeled by an iterated powerset $\mathcal{P}^n$.

Local n-level superhyperoperation. Fix $n = 2$ for concreteness (sets of sets of outcomes). For each $U \in \tau$, define

$$\star_U : \mathcal{F}(U) \times \mathcal{F}(U) \longrightarrow \mathcal{P}^2\big(\mathcal{F}(U)\big)$$

as follows: given two local states $a, b \in \mathcal{F}(U)$ (e.g., two guideline modules such as "sepsis bundle" and "heart-failure pathway" instantiated with the current data), the value $a \star_U b$ is the *family of feasible care-plan families* on U obtained by combining the modules. An element of $a \star_U b$ is itself a set $P \subseteq \mathcal{F}(U)$ representing one *plan family*, whose members are concrete feasible plans (states) such as:

- a medication plan with allowable dose/timing alternatives,
- an imaging plan with acceptable modality/scheduling alternatives,
- a monitoring plan with alternative escalation triggers.

Thus $a \star_U b \in \mathcal{P}^2(\mathcal{F}(U))$ encodes *alternatives of alternatives*: a set of plan-families, each plan-family being a set of concrete feasible plans.

n-level restriction-compatibility (scope consistency). If $V \subseteq U$ is a smaller scope (e.g., restricting a campus-wide plan to the ICU), then every global plan-family, restricted to V, must still be feasible as a plan-family computed directly on V:

$$(\rho_{U,V})^{(2)}\big(a \star_U b\big) \;\subseteq\; \rho_{U,V}(a) \star_V \rho_{U,V}(b).$$

Operationally: any hierarchy of feasible care-plan alternatives produced at a broad scope, when projected to a subscope, remains within the hierarchy of feasible alternatives available at that subscope. Hence the decision-support workflow forms a *Sheaf SuperHyperStructure* that encodes hierarchical multi-valued clinical composition consistently from local units to global hospital scopes.

Theorem 10.2.3 (Well-definedness of n-Sheaf SuperHyperStructures)**.** *Let (X, τ) be a topological space and let $\mathcal{F}$ be a sheaf of sets on X with restriction maps $\rho_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$ for $V \subseteq U$. Fix $n \in \mathbb{N}$ and define the iterated powerset functor $\mathcal{P}^n$ by*

$$\mathcal{P}^0(S) := S, \qquad \mathcal{P}^{k+1}(S) := \mathcal{P}\big(\mathcal{P}^k(S)\big) \quad (k \geq 0).$$

For any map $f : A \to B$, define inductively the induced maps

$$f^{(0)} := f, \qquad f^{(k+1)} := \big(f^{(k)}\big)^\sharp : \mathcal{P}^{k+1}(A) \to \mathcal{P}^{k+1}(B),$$

where $g^\sharp : \mathcal{P}(\mathrm{dom}(g)) \to \mathcal{P}(\mathrm{cod}(g))$ denotes direct image: $g^\sharp(S) := \{g(s) \mid s \in S\}$.

Assume that for each open set $U \in \tau$ we are given an n-level superhyperoperation

$$\star_U : \mathcal{F}(U) \times \mathcal{F}(U) \longrightarrow \mathcal{P}^n\big(\mathcal{F}(U)\big),$$

*and that the n-*level restriction-compatibility *condition holds: for every inclusion $V \subseteq U$ and all $a, b \in \mathcal{F}(U)$,*

$$(\rho_{U,V})^{(n)}\big(a \star_U b\big) \;\subseteq\; \rho_{U,V}(a) \star_V \rho_{U,V}(b). \tag{10.4}$$

Then the data $(\mathcal{F}, \{\star_U\}_{U \in \tau})$ is well-defined in the following sense:

1. *For every $U \in \tau$, the map $\star_U$ is well-typed and defines an n-level superhyperoperation on the carrier set $\mathcal{F}(U)$, i.e., $\star_U : \mathcal{F}(U) \times \mathcal{F}(U) \to \mathcal{P}^n(\mathcal{F}(U))$.*

2. *For every inclusion $V \subseteq U$, the restriction map $\rho_{U,V}$ is a* (weak) homomorphism of n-level superhyperstructures *in the sense of* (10.4).

3. *The compatibility is* coherent under iterated restriction*: for any $W \subseteq V \subseteq U$ and any $a, b \in \mathcal{F}(U)$, one has*

$$(\rho_{U,W})^{(n)}\big(a \star_U b\big) \;\subseteq\; \rho_{U,W}(a) \star_W \rho_{U,W}(b),$$

so the notion does not depend on whether restriction is applied in one step or in several steps.

Proof. **(1)** Fix $U \in \tau$. By assumption, $\star_U : \mathcal{F}(U) \times \mathcal{F}(U) \to \mathcal{P}^n(\mathcal{F}(U))$ is a well-typed map into the n-fold iterated powerset of the carrier set $\mathcal{F}(U)$. Thus $\star_U$ is an n-level superhyperoperation on $\mathcal{F}(U)$ by definition.

(2) Let $V \subseteq U$. The restriction map $\rho_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$ is a well-defined function from the sheaf data. By the inductive construction of $f^{(n)}$, the map

$$(\rho_{U,V})^{(n)} : \mathcal{P}^n(\mathcal{F}(U)) \longrightarrow \mathcal{P}^n(\mathcal{F}(V))$$

is also well-defined. Condition (10.4) is precisely the required (weak) homomorphism property for $\rho_{U,V}$ between the local n-level superhyperstructures.

(3) We establish coherence under iterated restriction.

Step 1: Functoriality of the iterated direct image. For maps $A \xrightarrow{f} B \xrightarrow{g} C$, we claim

$$(g \circ f)^{(n)} = g^{(n)} \circ f^{(n)} \qquad \text{and} \qquad (\mathrm{id}_A)^{(n)} = \mathrm{id}_{\mathcal{P}^n(A)}. \tag{10.5}$$

We prove the first identity by induction on n (the second is similar). For $n = 0$, it is the ordinary functoriality of functions: $(g \circ f)^{(0)} = g \circ f = g^{(0)} \circ f^{(0)}$. Assume (10.5) holds for $n = k$. Then, using the definition $h^{(k+1)} = (h^{(k)})^\sharp$ and the elementary identity

$$(g^\sharp \circ f^\sharp)(S) = g^\sharp\big(f^\sharp(S)\big) = (g \circ f)^\sharp(S) \qquad (S \subseteq A),$$

we obtain

$$(g \circ f)^{(k+1)} = \big((g \circ f)^{(k)}\big)^\sharp = \big(g^{(k)} \circ f^{(k)}\big)^\sharp = \big(g^{(k)}\big)^\sharp \circ \big(f^{(k)}\big)^\sharp = g^{(k+1)} \circ f^{(k+1)}.$$

Thus (10.5) holds for all n.

Step 2: Coherence. Take open sets $W \subseteq V \subseteq U$ and elements $a, b \in \mathcal{F}(U)$. Since $\mathcal{F}$ is a sheaf (hence a presheaf), its restriction maps satisfy

$$\rho_{U,W} = \rho_{V,W} \circ \rho_{U,V}.$$

Applying Step 1 with $f = \rho_{U,V}$ and $g = \rho_{V,W}$ yields

$$(\rho_{U,W})^{(n)} = (\rho_{V,W})^{(n)} \circ (\rho_{U,V})^{(n)}. \tag{10.6}$$

By (10.4) for $V \subseteq U$,

$$(\rho_{U,V})^{(n)}\big(a \star_U b\big) \;\subseteq\; \rho_{U,V}(a) \star_V \rho_{U,V}(b).$$

Apply $(\rho_{V,W})^{(n)}$ to both sides. Since each $f^{(n)}$ is built from direct images on powersets, it is monotone with respect to $\subseteq$; hence

$$(\rho_{V,W})^{(n)}\Big((\rho_{U,V})^{(n)}(a \star_U b)\Big) \;\subseteq\; (\rho_{V,W})^{(n)}\Big(\rho_{U,V}(a) \star_V \rho_{U,V}(b)\Big).$$

Using (10.6), the left-hand side equals $(\rho_{U,W})^{(n)}(a \star_U b)$. On the right-hand side, apply (10.4) for $W \subseteq V$ with $a' = \rho_{U,V}(a)$ and $b' = \rho_{U,V}(b)$ to obtain

$$(\rho_{V,W})^{(n)}\Big(\rho_{U,V}(a) \star_V \rho_{U,V}(b)\Big) \;\subseteq\; \rho_{V,W}\big(\rho_{U,V}(a)\big) \star_W \rho_{V,W}\big(\rho_{U,V}(b)\big).$$

Finally, $\rho_{V,W} \circ \rho_{U,V} = \rho_{U,W}$, so the right-hand side becomes $\rho_{U,W}(a) \star_W \rho_{U,W}(b)$. Combining these inclusions yields

$$(\rho_{U,W})^{(n)}\big(a \star_U b\big) \;\subseteq\; \rho_{U,W}(a) \star_W \rho_{U,W}(b),$$

which is exactly the desired coherence under iterated restriction.

Therefore, the definition of an n-sheaf superhyperstructure is internally consistent and well-defined. □

Furthermore, we present several concrete examples of an n-sheaf superhyperstructure.

Example 10.2.4 (A constant-type (trivial) sheaf SuperHyperStructure)**.** Let $(X, \mathcal{O}_X)$ be a topological space and fix the base set $\mathbb{Z}$. Consider the *constant sheaf* $\underline{\mathbb{Z}}$ defined by

$$H(U) := \underline{\mathbb{Z}}(U) = \{\, s : U \to \mathbb{Z} \mid s \text{ is locally constant} \,\}, \qquad U \in \mathcal{O}_X,$$

with restriction maps $\rho_{V,U}(s) = s|_V$ for $V \subseteq U$.

(Sheaf axiom). If $U = \bigcup_i U_i$ and $s_i \in H(U_i)$ satisfy $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i, j, then the glued map $s : U \to \mathbb{Z}$ given by $s(x) := s_i(x)$ for $x \in U_i$ is well-defined, locally constant, and satisfies $s|_{U_i} = s_i$; moreover, it is unique.

(A $(1,2)$–superhyperoperation on each open set). For each open U, define $\star_U : \mathcal{P}(H(U)) \times \mathcal{P}(H(U)) \to \mathcal{P}^2(H(U))$ by

$$\star_U(A, B) := \big\{\, A \oplus_U B,\; A \ominus_U B \,\big\},$$

where

$$A \oplus_U B := \{\, a + b \mid a \in A,\; b \in B \,\}, \qquad A \ominus_U B := \{\, a - b \mid a \in A,\; b \in B \,\},$$

and $(a \pm b)(x) := a(x) \pm b(x)$ pointwise. Since sums/differences of locally constant maps are locally constant, $A \oplus_U B$ and $A \ominus_U B$ are subsets of $H(U)$, hence $\star_U(A, B) \in \mathcal{P}^2(H(U))$.

(Restriction compatibility). For every inclusion $V \subseteq U$ and all $A, B \subseteq H(U)$,

$$\rho^{[2]}_{V,U}\big(\star_U(A, B)\big) = \star_V\big(\rho^{[1]}_{V,U}(A), \rho^{[1]}_{V,U}(B)\big),$$

because $(a \pm b)|_V = a|_V \pm b|_V$ implies $\rho^{[1]}_{V,U}(A \oplus_U B) = (\rho^{[1]}_{V,U}A) \oplus_V (\rho^{[1]}_{V,U}B)$ and similarly for $\ominus$. Therefore $\big(U \mapsto (H(U), \star_U)\big)$ forms a concrete *Sheaf SuperHyperStructure* (of profile $\Theta = (1, 2, 2)$) which is "constant-type" in the sense that it is built from the constant sheaf $\underline{\mathbb{Z}}$.

Example 10.2.5 (Gluing local data into a global section (continuous case))**.** Let $X = [0,1]$ with the usual topology and let

$$H(U) := C(U, \mathbb{R})$$

be the set of continuous real-valued functions on U, with restriction maps $\rho_{V,U}(f) = f|_V$ for $V \subseteq U$. It is standard that $U \mapsto C(U, \mathbb{R})$ is a sheaf: compatible local continuous functions glue uniquely to a global continuous function.

Fix again $\Theta = (1, 2, 2)$ and define, for each open $U \subseteq X$,

$$\star_U(A, B) := \{\, A \oplus_U B,\ A \otimes_U B \,\} \in \mathcal{P}^2(H(U)),$$

where for $A, B \subseteq H(U)$,

$$A \oplus_U B := \{\, a + b \mid a \in A,\ b \in B \,\}, \qquad A \otimes_U B := \{\, a \cdot b \mid a \in A,\ b \in B \,\},$$

and $(a + b)(x) = a(x) + b(x)$, $(a \cdot b)(x) = a(x)b(x)$. Closure holds since sums/products of continuous functions are continuous.

Local-to-global gluing (explicit). Let

$$U_1 = (-0.1, 0.6) \cap [0,1], \qquad U_2 = (0.4, 1.1) \cap [0,1],$$

so that $X = U_1 \cup U_2$ and $U_{12} := U_1 \cap U_2 = (0.4, 0.6)$. Define local sections

$$f_1 \in C(U_1, \mathbb{R}), \quad f_1(x) = x, \qquad f_2 \in C(U_2, \mathbb{R}), \quad f_2(x) = x.$$

They agree on the overlap: $f_1|_{U_{12}} = f_2|_{U_{12}}$. Hence, by the sheaf axiom, there exists a unique global section $f \in C(X, \mathbb{R})$ such that $f|_{U_1} = f_1$ and $f|_{U_2} = f_2$, namely $f(x) = x$ on $[0,1]$.

Compatibility with the superhyperoperation. The restriction compatibility

$$\rho^{[2]}_{V,U}\big(\star_U(A, B)\big) = \star_V\big(\rho^{[1]}_{V,U}(A), \rho^{[1]}_{V,U}(B)\big)$$

holds for all $V \subseteq U$ since restriction commutes with $+$ and $\cdot$. Consequently, not only do individual local sections glue to a global section, but set-based local computations performed via $\star$ are consistent under restriction, which is the intended "sheaf" behavior for SuperHyperStructures.

Chapter 11

Conclusion

In this book, we investigated a broad range of mathematical and applied concepts that can be formulated within the framework of HyperStructures and SuperHyperStructures. By replacing single-valued operations with set-valued (hyper)operations and, moreover, by working on iterated powersets, these frameworks provide a unified language for modeling multiway interactions, nondeterministic outcomes, and hierarchical (nested) forms of uncertainty.

Looking forward, several research directions appear particularly promising. First, it is important to develop efficient *algorithmic* procedures for core tasks such as evaluation of superhyperoperations, propagation of set-valued information, and computation of invariants (e.g., connectivity- and centrality-type measures, or topological indices in superhypergraph settings). Second, we expect progress on *computational* and *quantitative* validation, including benchmark datasets, simulation studies, and sensitivity analyses that measure robustness under noisy or incomplete information. Such empirical investigations can clarify when hyper- and superhyper-extensions yield practical advantages over classical models and can guide principled choices of the profile parameters (m, n, k) in real applications.

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Use of Generative AI and AI-Assisted Tools

We use generative AI and AI-assisted tools for tasks such as English grammar checking, English translation, and troubleshooting LaTeX code errors, and we do not employ them in any way that violates ethical standards. The final results are reviewed and verified by the authors.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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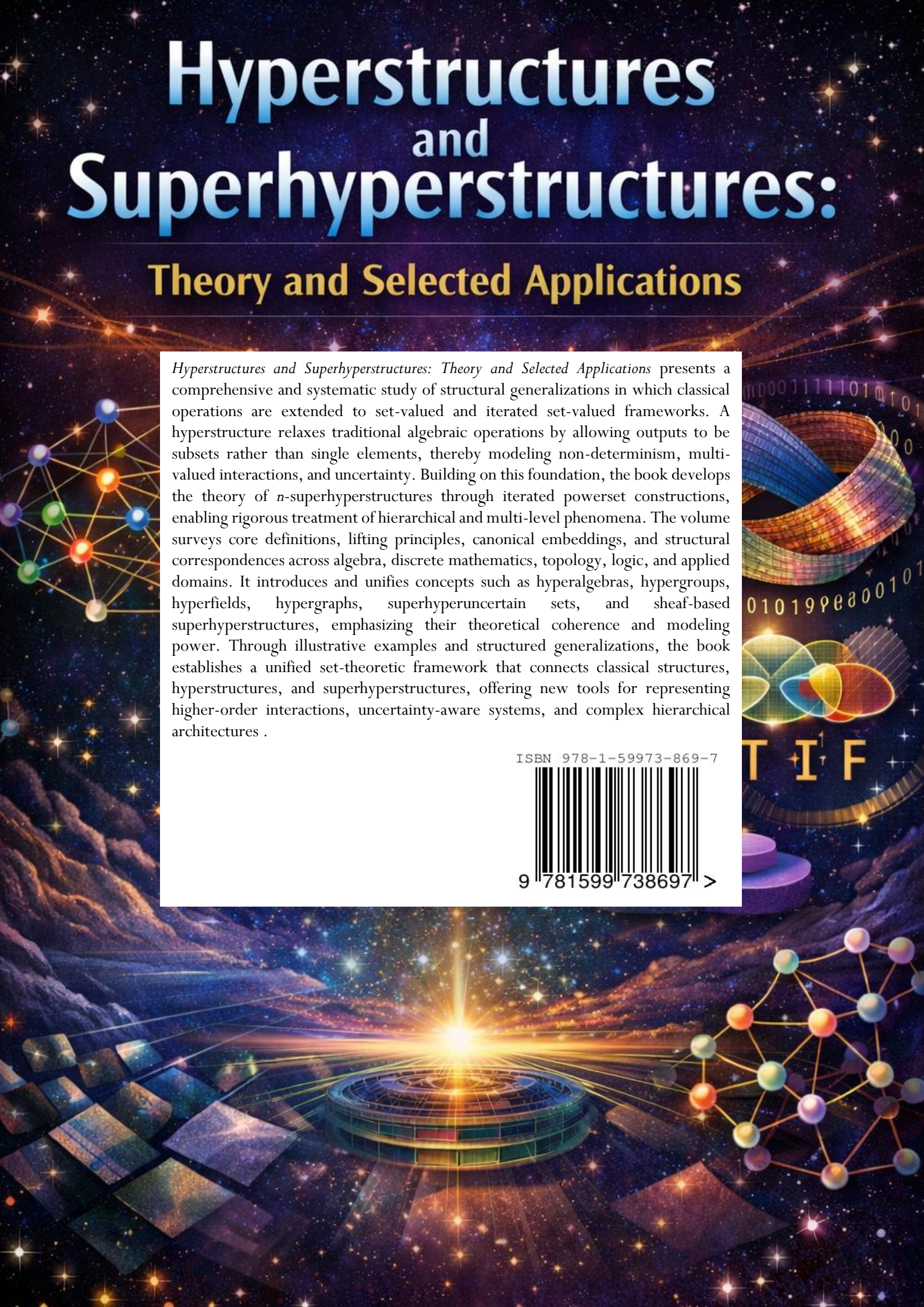
Hyperstructures and Superhyperstructures:

Theory and Selected Applications



Hyperstructures and Superhyperstructures: Theory and Selected Applications presents a comprehensive and systematic study of structural generalizations in which classical operations are extended to set-valued and iterated set-valued frameworks. A hyperstructure relaxes traditional algebraic operations by allowing outputs to be subsets rather than single elements, thereby modeling non-determinism, multi-valued interactions, and uncertainty. Building on this foundation, the book develops the theory of n-superhyperstructures through iterated powerset constructions, enabling rigorous treatment of hierarchical and multi-level phenomena. The volume surveys core definitions, lifting principles, canonical embeddings, and structural correspondences across algebra, discrete mathematics, topology, logic, and applied domains. It introduces and unifies concepts such as hyperalgebras, hypergroups, hyperfields, hypergraphs, superhyperuncertain sets, and sheaf-based superhyperstructures, emphasizing their theoretical coherence and modeling power. Through illustrative examples and structured generalizations, the book establishes a unified set-theoretic framework that connects classical structures, hyperstructures, and superhyperstructures, offering new tools for representing higher-order interactions, uncertainty-aware systems, and complex hierarchical architectures .

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